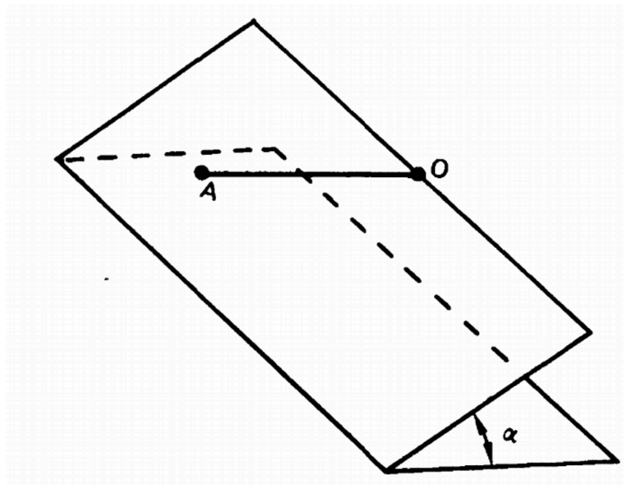
Problem 1. An incline plane

A pendulum initially swings with amplitude A (not necessarily small) on an incline, of inclination angle α with $\alpha \ll 1$. Determine how the amplitude depends on the inclination angle as it is slowly increased to $\alpha \sim 1$.



You should find $A_{\text{final}} \propto \frac{1}{(\sin\alpha)^{1/4}}$ for $\alpha \sim 1$.
Hint: Argue that the amplitude A will be small by the time α is large.

Problem 2. Short problems

Answer briefly. No more than a few lines

(a) Derive the canonical transformation rules $(q,p) \to (Q,P)$ for type $F_2(q,P,t)$

$$p = \frac{\partial F_2}{\partial q} \tag{1}$$

$$Q = \frac{\partial F_2}{\partial P} \tag{2}$$

$$H' = H + \frac{\partial F_2}{\partial t} \tag{3}$$

from the action principle. (This is essentially just reproducing what was done in lecture).

(b) It is well known that replacing the Lagrangian by

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{df}{dt}(q,t) \tag{4}$$

does not change the equations of motion. Show that this change in the Lagrangian amounts to a canonical transformation in the corresponding Hamiltonian setup, and find the generating function of type F_2 for this transformation.

(c) Consider the Hamiltonian for a particle in a electromagnetic field

$$H = \frac{(\boldsymbol{p} - e\boldsymbol{A})^2}{2m} + e\varphi(t,\boldsymbol{r}) \tag{5}$$

Under a gauge transformation the electromagnetic potentials $\boldsymbol{A}, \varphi$ change, but the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ do not. The change in the potentials is specified a function $\Lambda(t,\boldsymbol{r})$, with new potentials

$$\boldsymbol{A} \to \boldsymbol{A}'(t,\boldsymbol{r}) = \boldsymbol{A} + \nabla\Lambda(t,\boldsymbol{r}) \tag{6}$$

$$\varphi \to \varphi'(t,\boldsymbol{r}) = \varphi - \partial_t\Lambda(t,\boldsymbol{r}) \tag{7}$$

Show that this change in the Hamiltonian can be written as a canonical transformation, and find the corresponding F_2 generating function.

(d) (Optional but recommended) Spell out the relation between parts (c) and parts (b), by examining the Lagrangian for a particle in an electromagnetic field

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - e\varphi(t,\boldsymbol{r}) + \frac{e}{c}\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(t,\boldsymbol{r}) \tag{8}$$

(e) What is the transformation $(\boldsymbol{r},\boldsymbol{p}) \to (\boldsymbol{R},\boldsymbol{P})$ generated by $F_2(\boldsymbol{r},\boldsymbol{P}) = a\boldsymbol{r}\cdot\boldsymbol{P}$. Describe this transformation qualitatively.

(f) The Hamiltonian of a charged particle of charge q in the electrostatic potential of an electric dipole with dipole moment d_0 directed along the z axis is

$$H = \frac{\boldsymbol{p}^2}{2m} + \kappa \frac{\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}}}{r^2} \tag{9}$$

where $\kappa = q d_0 / (4\pi\epsilon_0)$ in SI units.

Use the previous item with $a = (1+\epsilon)$ to show that said particle has

$$\boldsymbol{p} \cdot \boldsymbol{r} - 2Et = \text{const} \tag{10}$$

Problem 3. Discretizing canonical transforms (Roceck)

Many physical systems are described by Hamiltonians which give rise to equations of motion that cannot be solved analytically, but must be discretized and solved numerically. Discretizations which preserve the symmetries of the continuum theory are especially effective when numerically integrating the equations of motion for long times. In this problem, we will explore some of the techniques available to describe such systems.

Consider a one-dimensional classical system whose s *finite* time evolution is described by a canonical transformation. Specifically, let

$$x_0 \equiv x(0) \quad , \quad x \equiv x(t) \quad , \quad p_0 \equiv p(0) \quad , \quad p \equiv p(t)$$

and consider a generating function $F_2(x_0, p, t)$. Then the evolution from (x_0, p_0) to (x, p) is obtained by solving the equations

$$p_0 = \frac{\partial F_2}{\partial x_0} \quad , \quad x = \frac{\partial F_2}{\partial p} \tag{11}$$

We are thinking of t as being small but finite.

(a) (i) Show that this evolution preserves volume in phase space (that is, prove Liouville's theorem for this case).

(ii) Next show that for

$$F_2(x_0, p, t) = x_0 p + t\, H(x_0, p)$$

as $t \to 0$, the evolution equations reduce to Hamilton's equations of motion.

(b) For a Hamiltonian of the form $\frac{p^2}{2m}+U(x)$, show that the naive discretization of Newton's equations of motion (for δt small but finite)

$$p = p_0 - \frac{\partial U(x_0)}{\partial x_0} t \quad , \quad x = x_0 + \frac{p_0}{m} t \tag{12}$$

does NOT preserve volume in phase space. For a harmonic oscillator, will the volume shrink or grow? What does this say about the long time behavior of this approximation? Estimate the number of iterations of this map before the error is of order one, in terms of the mass m of the particle, the spring constant k, and the finite interval t.

(c) What is the analogous discretization using the canonical transformations as outlined in a.ii? Work out the equations corresponding to (2) in part (c). Why is this *guaranteed* to preserve volume in phase space?

Recall that under a time dependent canonical map from $(q_1, p_1) \to (Q, P)$ generator $F_2(q_1, P, t)$ we have

$$p_1 = \frac{\partial F_2}{\partial q} \tag{13}$$

$$Q = \frac{\partial F_2}{\partial P} \tag{14}$$

$$H'(Q, P) = H(q_1, p_1) + \frac{\partial F_2(q_1, P, t)}{\partial t}\,. \tag{15}$$

This last part studies the implications of the last equation relating H' and H for discretization, and its meaning more generally.

We are describing a canonical map from $(x_0, p_0) \to (x, p)$. The Hamiltonian for (x, p) is $p^2/2m + U(x)$ so that the exact time evolution of the coordinates at time t is the differential equation we are trying to solve

$$\dot{x} = \frac{p}{m} \tag{16}$$

$$\dot{p} = -\frac{\partial U(x)}{\partial x} \tag{17}$$

These equations determine $x_+ = x(t+\delta t)$ and $p_+ = p(t+\delta t)$ for some *infinitessimal* δt. x_+, p_+ are not the same as taking (x_0, p_0) and applying the map generated by $F_2(x_0, p_+, t + \delta t)$. However, if we evolve x_0, p_0 with a new Hamiltonian H_0

$$\dot{x}_0 = \frac{\partial H_0(x_0, p_0)}{\partial p_0} \tag{18}$$

$$\dot{p}_0 = -\frac{\partial H_0(x_0, p_0)}{\partial x_0} \tag{19}$$

by infinitessimal δt to $x_{0+} = x_0(\delta t) = x_0 + \delta x_0$ and $p_{0+} = p_0(\delta t) = p_0 + \delta p_0$, and then apply the map generated by $F_2(x_{0+}, p_+, t + \delta t)$ to x_{0+}, p_{0+} we will exactly obtain (x_+, p_+). This is the meaning of a time dependent canonical transform, we can view the evolution either with x_0, p_0 or x, p. Ideally the time evolutions of x_0, p_0 will be approximately zero if the map $F_2(x_0, P, t)$ is a good approximation for the onshell action (principal function), $\underline{S}_2(t, q, t_0, P)$.

(d) (Optional but highly receommended) Compute H_0 using by two methods: (i) by using an appropriate version of Eq. (13), and (ii) by determining what H_0 needs to be so that the map generated by $F_2(x_{0+}, p_+, t + \delta t)$, maps (x_{0+}, p_{0+}) to (x_+, p_+)

You should find by both methods that

$$H_0(q_0, p_0) \approx t \frac{\partial U(q_0)}{\partial q} \frac{p_0}{m} + O(t^2) \tag{20}$$

Here H_0 is non-zero to first order in t, and is therefore small. For a second order symplectic integrator one would find $H_0 = 0 + O(t^2)$. See Ruth, IEEE Transactions on Nuclear Science (posted online).