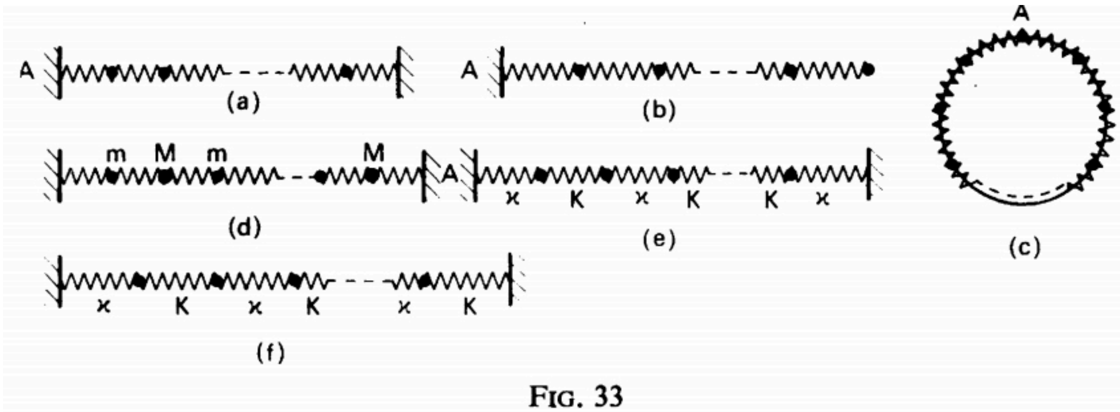

Figure 1: A number of figures for the problems on springs

Problem 1. Group velocity of a chain from a continuum theory

(a) Determine the frequencies of the eigen-vibrations of a system of $2N$ particles, alternating with masses m and M, connected by springs of elastic constant γ and separation a. This is similar to the problem shown in Fig. 33d above, but we will we assume periodic boundary conditions, $q_N(t) = q_{-N}(t)$

Hint: Try an ansatz

$$q_j = \xi_1 e^{i(kx_j - \omega t)} \tag{1}$$

$$q_{j+1} = \xi_2 e^{i(kx_{j+1} - \omega t)} \tag{2}$$

and find a two-by-two eigen value equation for (ξ_1, ξ_2). This gives two eigen frequencies $\omega_\pm(k)$ for each value of k.

(b) Determine the dispersion curve $\omega_\pm(k)$ at small k, $ka \ll 1$, to order k^3 and sketch $\omega_+(k)$ and $\omega_-(k)$ at small k on the same graph. Determine the group velocity to order k^2.

(c) When the wavelength of the waves of part (a) is very long, the microscopic details of the discrete model in (a), are unimportant. A continuum theory can reproduce the results of the model in (a), provided the "low energy" constants of the continuum theory are adjusted to match certain physical properties.

Consider the action from a previous problem

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha(\partial_x^2 q)^2 \tag{3}$$

From the equation of motion you found previously, determine the dispersion curve $\omega(k)$ associated with this action. What should the values of the "low-energy constants", μ, T and α be set to if the continuum action in Eq. (3), is to reproduce the dispersion curve of the discrete theory of parts (a) and (b) at small k for the "plus" modes (i.e. the modes with eigenfrequencies $\omega_+(k)$ in the discrete theory) and how should they be tuned to reproduce the "minus" modes.

Problem 2. Split personality

(a) A general solution to the wave equation is

$$y(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[A(k)e^{i(kx-\omega(k)t)} + B(k)e^{i(kx+\omega(k)t)}\right] \tag{4}$$

where $\omega(k)$ is a positive symmetric function of k, $\omega(-k) = \omega(k)$. For a real wave $B(-k)$ must be equal to $A^*(k)$. By change of variables $k \to -k$ in the second integral the solution can be written[1]

$$y(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[A(k)e^{i(kx-\omega(k)t)} + A^*(k)e^{-i(kx-\omega(k)t)}\right] \tag{5}$$

The wave equation is a second order differential equation. Thus in order to specify the problem, we need to specify the initial amplitude $y(0,x)$ and the initial velocity $\partial_t y(0,x)$ everywhere on the string. How is $A(k)$ determined by $y(0,x)$ and $\partial_t y(0,x)$?

(b) Here we want to describe a wave-packet which moves to the right. The amplitude at $t = 0$ is

$$y(0,x) = \mathrm{Re}[g(x)e^{ik_0 x}]\,, \tag{6}$$

with

$$g(x) = \frac{1}{\sqrt{2\pi a^2}} \exp(-x^2/(2a^2))\,, \tag{7}$$

and $k_0 a \gg 1$. Argue that the appropriate initial condition for a right moving wave is

$$\partial_t y \simeq v_\phi\, \partial_x y \tag{8}$$

where $v_\phi(k_0) = \omega(k_0)/k_0$ is the phase velocity of the wave, by (approximately) computing $A(k)$ in this case. What would $A(k)$ be if $\partial_t y(0,x) = 0$? Sketch $|A(k)|^2$ in both cases.

In the second case $\partial_t y = 0$, one can either calculate the result directly or use the superposition principle.

(c) Repeat the argument (given in class for complex waves) that if the solution for a wave is

$$y(t,x) = \int \frac{dk}{2\pi} \left[A(k)e^{i(kx-\omega(k)t)} + A^*(k)e^{-i(kx-\omega(k)t)}\right]\,, \tag{9}$$

then, provided the wave form is initialized as in (8), then

$$y(t,x) \simeq \cos(k_0 x - \omega_0 t)\, g(x - Ut)\,. \tag{10}$$

Here $U = d\omega(k_0)/dk$ is the group velocity and $\omega_0 = \omega(k_0)$. The applet by Michael Fowler is a helpful visualization.

(d) Determine the wave form at late times if $\partial_t y(0,x) = 0$. Hint: use the superposition principle.

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[1] Having had this discussion with the grad-students in the past ... $\int_{-\infty}^{\infty} dk f(k) = \int_{+\infty}^{-\infty} -d\tilde{k} f(-\tilde{k}) = \int_{-\infty}^{\infty} d\tilde{k} f(-\tilde{k})$, and then since $\tilde{k}$ is a dummy integration variable, we now just call it k to arrive at the result Eq. (5)