## Problem 1. Transmitted and reflected wave forms

A string is divided into two parts with mass density $\mu_1$ on the left, and mass density $\mu_2$ on the right. Consider a wave form, $y_{\rm in}(x - v_1 t)$, traveling from the left to the right, which scatters off the change in density (or impedance).

(a) Show that when $y_{\rm in}(t, x)$ is a traveling plane wave, $A\, e^{ikx-wt}$, the reflected and transmitted waves respectively take the form:

$$B = A\,\tilde{r},\, e^{i(-kx-\omega t)}\,, \tag{1}$$

$$C = A\,\tilde{t}\, e^{i(k'x-\omega t)}\,, \tag{2}$$

with $k' = \omega/v_2$, and

$$r = -\frac{Z_2 - Z_1}{Z_1 + Z_2}\,, \tag{3}$$

$$t = \frac{2Z_1}{Z_1 + Z_2}\,. \tag{4}$$

where the wave impedance is $Z = \sqrt{T\mu}$.

(b) Continue working with the plane waves of part ($a$). Show that the energy transported per time by the incoming wave plane wave of part ($a$), equals the sum of the energies transported per time of the outgoing plane waves (i.e. the reflected and transmitted waves).

(c) Now take a general real wave form $y_{\rm in}(x-vt)$, and assume that the wave form $y_{\rm in}(x-vt)$ has no appreciable support after a distance $L$, i.e. $y_{\rm in}(x) \simeq 0$ for $|x| > L$. Determine the wave form of the transmitted and reflected waves in coordinate space in terms of $y_{\rm in}(x - vt)$.

Show by integration of these forms that at late times $t \gg L/v$ the total energy in the transmitted and reflected waves equals the energy in the incoming wave.

**Solution:**

(a) The general solution on the left is with harmonic time dependence $e^{-i\omega t}$ is

$$y_L(t,x) = Ae^{ikx-i\omega t} + Be^{-ikx-\omega t} \tag{5}$$

with $k = \omega/v_1 \propto \mu_1$ with $v_1 = \sqrt{T/\mu_1}$

The general solution on the right with harmonic time dependence is

$$y_R(t,x) = Ce^{ik_2x-i\omega t} + De^{-ik_2x-\omega t} \tag{6}$$

with $k_2 = w/v_2 \propto \sqrt{\mu_2}$.

The frequency dependence of the two waves must be the same in order to satisfy the boundary conditions (i.e. that the string is continuous) at all times. The tension is the same on both sides, otherwise the midlle point would move to the right or left.

There is no wave coming from the right and one can set $D = 0$. At the middle point we have continuity

$$A + B = C \tag{7}$$

We also have a continuity of derivative

$$T\frac{\partial y}{\partial x}\bigg|_L = T\frac{\partial y}{\partial x}\bigg|_R \tag{8}$$

which says that (if $dy/dx$ is positive) that the upward force by the right balances the downward force by the left, otherwise the middle point would have a net force leading (since it has negligible mass) to infinite accelearation.

$$TA(ik) + TB(-ik) = TC(ik_2) \tag{9}$$

Solving these equations equation using $k_2 = \sqrt{\mu_2/\mu_1}k$ leads to the desired result

$$\tilde{r} = \frac{B}{A} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{10}$$

$$\tilde{t} = \frac{C}{A} = \frac{2Z_1}{Z_1 + Z_2} \tag{11}$$

with $Z = \sqrt{T\mu}$.

(b) Here we simply use the expression for the Poynting flux from lecture.

$$S^x = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \tag{12}$$

For a plane wave $e^{ikx-i\omega t}$ this evaluates to

$$\overline{S}^x = \frac{Z}{2}\omega^2|A|^2 \tag{13}$$

For a reflected wave $e^{-ikx-i\omega t}$ this evaluates to

$$\overline{S}^x = -\frac{Z}{2}\omega^2|A|^2 \tag{14}$$

Drawing the pillbox around the central point (see figure) we should have

$$\overline{S}^x_A + \overline{S}^x_B = \overline{S}^x_C \tag{15}$$

Leading to a requirement that

$$|A|^2 = |B|^2 + |C|^2\frac{Z_2}{Z_1} \tag{16}$$

i.e.

$$1 = \tilde{r}^2 + \tilde{t}^2\left(\frac{Z_2}{Z_1}\right) \tag{17}$$

So to verify energy conservation we need that

$$1 = \frac{(Z_1 - Z_2)^2}{(Z_1 + Z_2)^2} + \frac{4Z_1Z_2}{(Z_1 + Z_2)^2} \tag{18}$$

which obviously holds.

(c) By assumption

$$y_A(t,x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}A(k)e^{ik(x-vt)} \tag{19}$$

Instead of integrating over $k$ we can integrate over frequency $\omega = vk$. Changing variables

$$y_A(t,x) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\hat{A}(\omega)\,e^{i\omega((x/v)-t)} \tag{20}$$

where $\hat{A}(\omega) = A(k)/v$. Let us denote

$$y_0(x) = y_A(0,x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}A(k)e^{ikx} \tag{21}$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{A}(\omega)e^{i(\omega/v)x} \tag{22}$$

which is the shape of the undisturbed incoming wave at a fixed time ($t = 0$).

The incoming wave is

$$y_A(t,x) = y_0(x - vt) \tag{23}$$

The reflected wave is

$$y_B(t,x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}B(k)e^{ik(-x-vt)} \tag{24}$$

The transmitted wave takes the form

$$y_C(t,x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}C(k)e^{ik(x-v_2t)} \tag{25}$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{C}(\omega)e^{i\omega((x/v_2)-t)} \tag{26}$$

Our fourier analysis of $(a)$ shows that

$$\hat{B}(\omega) = \tilde{r}\hat{A}(\omega) \tag{27}$$

And the relation between the amplitudes at the same *frequency* reads

$$\hat{C}(\omega) = \tilde{t}\hat{A}(\omega) \tag{28}$$

This follows because the boundary conditions are specified at all times at $x = 0$. The incoming wave, the reflected wave, and the transmitted wave at $x = 0$ and arbitrary time take the form

$$y_A(t,0)|_{x=0} = \int \frac{d\omega}{2\pi}\hat{A}(\omega)e^{-i\omega t} \tag{29}$$

$$y_B(t,0)|_{x=0} = \int \frac{d\omega}{2\pi}\hat{B}(\omega)e^{-i\omega t} \tag{30}$$

$$y_C(t,0)|_{x=0} = \int \frac{d\omega}{2\pi}\hat{C}(\omega)e^{-i\omega t} \tag{31}$$

Requiring continuity for instance reads

$$y_A(t,0) + y_B(t,0) = y_C(t,0) \tag{32}$$

which will follow if

$$\hat{A}(\omega) + \hat{B}(\omega) = \hat{C}(\omega) \tag{33}$$

which should be compared to Eq. (7).

Thus the reflected wave is simply

$$y_{\rm B}(t,x) = \tilde{r}\; y_0(-x - vt) \tag{34}$$

while the transmitted wave form is

$$y_{\rm C}(t,x) = \tilde{t}\; \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)}\hat{A}\,(\omega)\, e^{i\omega((x/v_2)-t)} \tag{35}$$

Pulling out a factor of $v$

$$y_{\rm C}(t,x) =\tilde{t}\; \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)}\hat{A}\,(\omega)\, e^{i(\omega/v)(v(x/v_2)-t)} \tag{36}$$

$$=\tilde{t}\; y_0\left(\frac{v}{v_2}(x - v_2 t)\right) \tag{37}$$

Then we can integrate the energy in these waves. The energy density is

$$\epsilon = \frac{1}{2}\mu(\partial_t y)^2 + \frac{1}{2}T(\partial_x y)^2 \tag{38}$$

Then note that for any wave which moves entirely in one direction

$$\mp\frac{1}{v}\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} \tag{39}$$

where the minus sign is for a right moving wave $(x - vt)$ and the plus sign is for a left moving wave $(-x - vt)$. Thus for a right mover or a leftmover (but not for superposition)

$$\mu(\partial_t y)^2 = T(\partial_x y)^2 \tag{40}$$

and thus

$$\epsilon = T(\partial_x y)^2 \tag{41}$$

The total energy density in a wave which is either right or left moving is for example

$$E = \int_{-\infty}^{\infty} dx\, T\, (\partial_x y(\pm x - vt))^2 \tag{42}$$

$$= \int_{-\infty}^{\infty} du\, T\, (\partial_u y(u))^2 \tag{43}$$

where here and below $u$ is a dummy integration variable short for $u = \pm x - vt$

We look well before and after the reflection so that the waves consist of well separated either right or left movers. The total energy in $A$ is thus

$$E_A = \int_{-\infty}^{\infty} du\, T\, (\partial_u y_0(u))^2 \tag{44}$$

The energy in $B$ is similarly

$$E_B = \int_{-\infty}^{\infty} du\, \tilde{r}^2\, T(\partial_u y_0(u))^2 \tag{45}$$

The energy in $C$ is

$$E_C = \int_{-\infty}^{\infty} dx\, \tilde{t}^2\, T \left[ \frac{\partial}{\partial x} y_0 \left( \frac{v}{v_2}(x - v_2 t) \right) \right]^2 \tag{46}$$

Now it is convenient to define

$$u = \frac{v}{v_2}(x - v_2 t) \tag{47}$$

and then accounting for the change in measure $dx \to du$ etc, we have

$$E_C = \int_{-\infty}^{\infty} du\, \tilde{t}^2\, \frac{v}{v_2}\, T\, [\partial_u y_0\,(u)]^2 \tag{48}$$

$$= \int_{-\infty}^{\infty} du\, \tilde{t}^2\, \frac{Z_2}{Z_1}\, T\, [\partial_u y_0\,(u)]^2 \tag{49}$$

where in passing to the second line we have used $v/v_2 = Z_2/Z_1$. We see that since

$$1 = \tilde{r}^2 + \tilde{t}^2 \frac{Z_2}{Z_1} \tag{50}$$

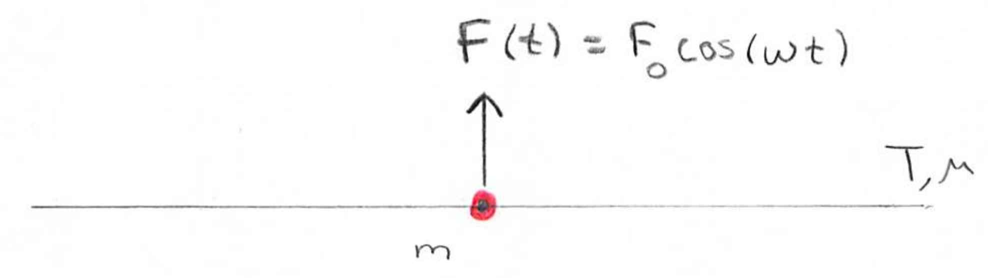
we have

$$E_A = E_B + E_C \tag{51}$$

as could be hoped.

## Problem 2. Oscillations on a string

Consider a long string of mass density $\mu$ and tension $T$ running along the $x$-axis. Attached to the center of the string is a mass, $m$, as shown below (see the last page). The mass is pushed and pulled by an an external harmonic force, $F(t) = F_0 \cos(\omega t) = \mathrm{Re}[F_0 e^{-i\omega t}]$, in the $y$ direction.




The amplitude of the strings motion $y(t,x) \propto e^{-i\omega t}$ is small enough to be treated in a harmonic approximation.

(a) Determine the steady state solution, $y(t,x)$, for the amplitude of the string to the right and left of the mass.

(b) Compute the time averaged work done per time by the external force.

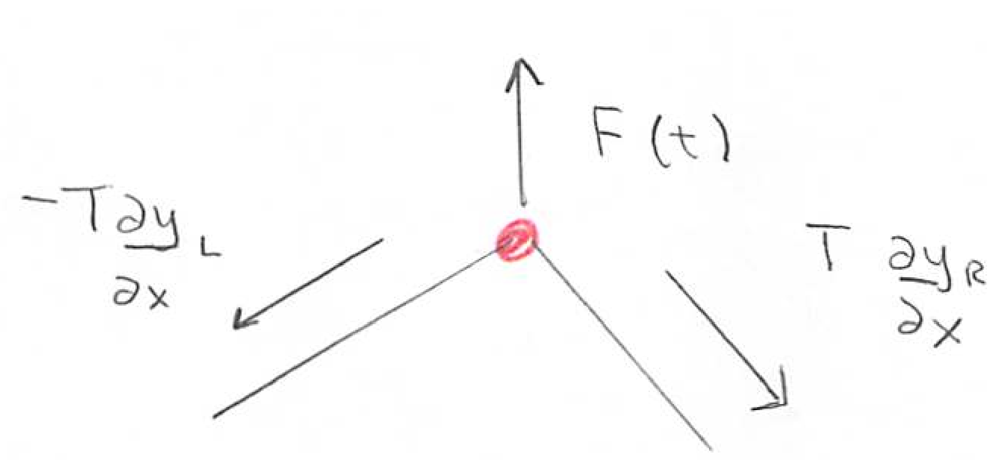



Figure 1: A free body diagram for the motion of the mass $m$.

**Solution:**

(a) The first part we recognize that the solution to right and the left must take the forms

$$y_L = Ae^{-ikx-i\omega t} \tag{52}$$
$$y_R = Be^{ikx-i\omega t} \tag{53}$$

Specifically the left solution consists only of left-movers, while the right solution consists only of right movers. Continuity demands that we set $A = B$. Drawing a free body digram of the mass (see below) which moves as $y_m(t)$ we determine a final constraint:

$$m\frac{d^2y_m}{dt^2} = F(t) + T\frac{dy_R}{dx} - T\frac{dy_L}{dx} \tag{54}$$

The mass moves up and down harmonically. Continuity says

$$y_m(t) = Ae^{-i\omega t} \tag{55}$$

Substituting the above forms into we find

$$-m\omega^2 A = F_0 + TikA - T(-ikA) \tag{56}$$

So since $Tik = iZ\omega$ with $Z = \sqrt{T\mu}$ the wave impedance we find

$$A = \frac{F_0}{-m\omega^2 + 2iZ\omega} \tag{57}$$

This completes the solution

$$\boxed{y(t,x) = \begin{cases} \frac{F_0}{-m\omega^2+2iZ\omega}e^{-ikx-i\omega t} & x < 0 \\ \frac{F_0}{-m\omega^2+2iZ\omega}e^{ikx-i\omega t} & x > 0 \end{cases}} \tag{58}$$

Taking the real part this can be written

$$\boxed{y(t,x) = \frac{F_0}{\sqrt{(m\omega)^2 + (2Z\omega)^2}} \cos(k|x| - vt + \phi_0) \qquad \phi_0 = \tan^{-1}(2Z\omega/m\omega^2)} \tag{59}$$

(b) The work done is

$$\frac{dW}{dt} = \dot{y}_m F(t) \tag{60}$$

Since these quantities vary harmonically as

$$y_m = Ae^{-i\omega t} \qquad \dot{y}_m = -i\omega A e^{-i\omega t} \qquad F(t) = F_0 e^{-i\omega t} \tag{61}$$

we can compute the average

$$\overline{\frac{dW}{dt}} = \frac{1}{2}\mathrm{Re}[(-i\omega A)\, F_0^*] \tag{62}$$

Substituting the amplitude from Eq. (57) into this last expression we find

$$\boxed{\overline{\frac{dW}{dt}} = \frac{1}{2}\mathrm{Re}\left[\frac{-i\omega F_0}{-m\omega^2 + 2iZ\omega} F_0^*\right] = \frac{|F_0|^2\omega^2 Z}{(m\omega^2)^2 + (2Z\omega)^2}} \tag{63}$$

## Problem 3. Time shift in a phase

A string of tension $T$ and mass per length $\mu$ is separated in two parts by a small ring of mass $m$. Consider the wave packet $y_{in}(x - vt)$ as described in the previous problem (problem 3), incident upon the ring. At $t = 0$ the center of the wave incoming wave packet will arrive at the ring, i.e. at negative times the wave the is centered around $-v|t|$.

(a) First consider a plane wave $A(k)\, e^{ikx-wt}$. Shown that the reflected and transmitted amplitudes are

$$B(k) = A(k)\, r(k)\, e^{i(-kx-\omega t)} \tag{64}$$

$$C(k) = A(k)\, t(k)\, e^{i(kx-\omega t)} \tag{65}$$

where

$$r(k) = -\frac{k}{k + ib} \tag{66}$$

$$t(k) = \frac{ib}{k + ib} \tag{67}$$

where $b = 2\mu/m$ has the units of wavenumber.

(b) Show that in general the reflected wave form is

$$y_{\rm trans}(t, x) = b \int_0^\infty d\xi\, e^{-b\xi}\, y_{\rm in}(x - vt + \xi) \tag{68}$$

$$y_{\rm refl}(t, x) = -\, y_{\rm in}(-(x + vt)) + b \int_0^\infty d\xi\, e^{-b\xi}\, y_{\rm in}(-(x + vt) + \xi) \tag{69}$$

(c) When the mass $m$ is very heavy, show that the reflection coefficient is

$$r(k) \simeq -e^{i\phi(k)} \tag{70}$$

where $\phi(k) \simeq -b/k$. The $k$-dependent phase has consequences.

Using the Fourier techniques of a previous problem, that a wavepacket incident on the ring (take the gaussian of the previous problem for definiteness), is reflected, and that that the center of the reflected wave packet is at

$$y_c = -v(t - t_0) \tag{71}$$

where the time delay is

$$t_0 = \frac{1}{v}\frac{d\phi}{dk} = +\frac{b}{v k_0^2} \tag{72}$$

Argue that this result also follows from the stationary phase condition.

**Solution:**
(a) See also lecture. The left solution with frequency $\omega$ is

$$y_L = Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t} \tag{73}$$

While the right solution is

$$y_R = Ce^{ikx-i\omega t} \tag{74}$$

The boundary conditions are, first, continuity at $x = 0$ at all times

$$A + B = C\,. \tag{75}$$

Second, we have $F^y_{\rm net} = ma^y(t)$ for the ring. Drawing a free body diagram for the ring

$$-T\frac{dy_L}{dx} + T\frac{dy_R}{dx} = m\ddot{y} \tag{76}$$

Since the height of the ring is $y_R(t, 0) = Ce^{-i\omega t}$ (or you could use $y_L(t, 0) = (A + B)e^{-i\omega t}$) we have

$$-TA(ik) + TB(ik) + T(ikC) = -m\omega^2 C \tag{77}$$

We solve these equations to find the expected result

$$r(k) = -\frac{k}{k+ib} \tag{78}$$

$$t(k) = \frac{ib}{k+ib} \tag{79}$$

(b) The incoming wave is of the form

$$y_A(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)e^{ik(x-vt)}\,, \tag{80}$$

$$= y_{\rm in}(x - vt)\,. \tag{81}$$

The reflected wave is

$$y_B(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} r(k)A(k)e^{ik(-x-vt)}\,. \tag{82}$$

Let us denote $u = -(x + vt)$

$$y_B(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} r(k)A(k)e^{iku}\,. \tag{83}$$

This then becomes a convolution

$$y_B(t, x) = \int_{-\infty}^{\infty} du' y_{\rm in}(u - u')r(u')\,, \tag{84}$$

$$= \int_{-\infty}^{\infty} du'' y_{\rm in}(u'')r(u - u'')\,, \tag{85}$$

where

$$r(u') = \int_k e^{iku'} \frac{-k}{k - ib} \,. \tag{86}$$

This integral is easy to do using the table of Fourier transforms

$$\begin{aligned} r(u') &= \int_k e^{iku'} \frac{-k}{k + ib} \,, & (87)\\ &= \int_k e^{iku'} \left( -1 + \frac{ib}{k + ib} \right) \,, & (88)\\ &= -\,\delta(u') + \theta(-u') b e^{bu'} \,. & (89) \end{aligned}$$

So

$$y_B(t,x) = -y_{\rm in}(u) + b \int_{-\infty}^{0} du'\, y_{\rm in}(u-u')\, e^{bu'} \tag{90}$$

Leading to

$$\boxed{y_B(t,x) = -y_{\rm in}(-(x+vt)) + b \int_0^\infty d\xi\, e^{-b\xi}\, y_{\rm in}(-(x+vt)+\xi)} \tag{91}$$

Similarly the transmitted wave is

$$y_C(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{t}(k) A(k) e^{iku} \tag{92}$$

but now $u = x - vt$. So

$$y_C(t,x) = \int_u y_{\rm in}(u-u') \tilde{t}(u') \tag{93}$$

Here

$$\begin{aligned} t(u') &= \int_k e^{iku'} \frac{ib}{k + ib} & (94)\\ &= \theta(-u') b e^{bu'} & (95) \end{aligned}$$

Leading to

$$y_C(t,x) = b \int_{-\infty}^{0} y_{\rm in}(u-u') e^{bu'} \tag{96}$$

or

$$\boxed{y_C(t,x) = b \int_0^\infty e^{-b\xi} y_{\rm in}((x-vt)+\xi)} \tag{97}$$

(c) The amplitude reflection coefficient is

$$\tilde{r}(k) = \frac{-k}{k+ib} \simeq -\frac{k}{k}(1 - i\frac{b}{k}) \simeq -e^{-i\phi(k)} \tag{98}$$

with $\phi(k) = b/k$. We note the phase in an odd function of $k$

$$y_R(t,x) = -\int \frac{dk}{2\pi} e^{ikx} A(k) e^{i\phi(k)} e^{ik(-x-vt)} \tag{99}$$

Then take the wave packet of the previous problem and expand near $k_0$ we have

$$\phi(k) + k(-x - vt) = \phi(k_0) - k_0(x + vt) + \tilde{k}(-(x + vt) + \phi'(k_0)) \tag{100}$$

Following the steps of the previous problem one

$$y_R(t,x) = -\cos(-k_0(x + vt) + \phi_0)\; g(-(x + vt) + \phi'(k_0)) \tag{101}$$

The center of the wave packet of the is given by the condition that the argument of $g$ is zero

$$-x_c(t) - vt + \phi'(k_0) = 0 \tag{102}$$

Solving

$$x_c(t) = -vt + \phi'(k_0) \tag{103}$$

So we see that the reflected wave is slightly *behind* (less negative) than the naive expectation, $-vt$, by $b/vk_0^2$.

$$\boxed{x_c(t) = -vt + \frac{b}{vk_0^2}} \tag{104}$$