Problem 1. (Milton, de Raad, Schwinger) Virial theorem from Noether logic

The virial theorem says that for the periodic motion of a particle the time averaged kinetic energy is related to an average of the potential energy:

$$\overline{2T} = \overline{\boldsymbol{r} \cdot \frac{\partial U(\boldsymbol{r})}{\partial \boldsymbol{r}}} \,. \tag{1}$$

For simplicity we will limit ourselves to the single particle Lagrangian

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - U(\boldsymbol{r}) \,, \tag{2}$$

but when many particles are involved, the theorem generalizes straightforwardly .

$$\overline{2T} = \overline{\sum_a \boldsymbol{r}_a \cdot \frac{\partial U(\boldsymbol{r})}{\partial \boldsymbol{r}_a}} \,. \tag{3}$$

Here we will derive this useful result using Noether logic[1].

Recall that we say that the trajectory is called "onshell" if it satisfies the equation of motion, and, when necessary, notate this by placing a bar underneath the coordinates $\underline{\boldsymbol{r}}(t)$

(a) For a closed orbit of potential $U(r) \propto r^\beta$ what is the statement of the virial theorem. What is the statement of the theorem for a harmonic oscillator $U(r) \propto r^2$ and the gravitational potential $U(r) \propto r^{-1}$.

(b) Consider a quantum mechanical particle in one dimension in an energy eigenstate $H |\psi_n(x)\rangle = E_n |\psi_n\rangle$ (an eigenstate is analogous to the classical periodic trajectory). Show that for this eigenstate we have

$$\langle 2T \rangle = \left\langle x \frac{\partial U(x)}{\partial x} \right\rangle \tag{4}$$

by considering $\langle\psi_n| [xp, H] |\psi_n\rangle$. (Incidentally we will see later in the course that the generator $G(x,p) = xp$ generates infinitesimal rescalings in the classical theory. That it why it is natural, see below, to consider the commutator $[G, H]$ in the quantum mechanical formulation.)

(c) Now return to classical mechanics. Consider a specific variation of the trajectory consisting of an infinitesimal rescaling of the coordinate $\boldsymbol{r}$

$$\boldsymbol{r} \to (1+\epsilon)\boldsymbol{r} \tag{5}$$

What is the change of the onshell action $S[\underline{\boldsymbol{r}}]$ for this specific variation over one complete period of a periodic classical trajectory $\underline{r}$?

(d) What is the change in the action $\delta S[\boldsymbol{r}, \delta\boldsymbol{r}]$ for the specific variation in Eq. (5). Do not assume that $\boldsymbol{r}$ is onshell.

(e) Using (c) and (d) prove the theorem in Eq. (1)

[1] It is not exactly the Noether theorem, since there is no conserved charge and no symmetry. But the derivation is essentially the same as is used to derive Noether theorem.

Problem 2. Foucault Pendulum and the Coriolis Effect (MIT-OCW)

(a) We showed in class using the Newtonian formalism that, in a rotating frame of reference, the equation of motion for a particle takes the form

$$m\boldsymbol{a}_r = \boldsymbol{F}_{\text{eff}} \,, \tag{6}$$

where

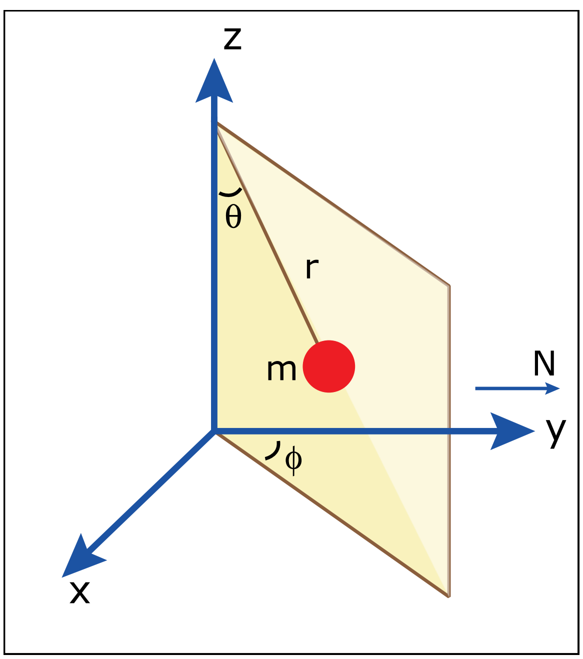
$$\boldsymbol{F}_{\text{eff}} = \boldsymbol{F} - 2m\,(\boldsymbol{\omega} \times \boldsymbol{v}_r) - m(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}))\,. \tag{7}$$

Here $\boldsymbol{v}_r = (dr^a/dt)\,\boldsymbol{e}_a(t)$ is the velocity in the rotating frame, and $\boldsymbol{e}_a(t)$ is the rotating basis of the frame. Derive this equation of motion from the Lagrangian formalism, where the Lagrangian in a fixed inertial frame is

$$L = \frac{1}{2}m\boldsymbol{v}^2 - U(\boldsymbol{r}) \tag{8}$$

with $\boldsymbol{v} = d\boldsymbol{r}/dt$.

Now consider a pendulum consisting of a long massless rod of length ℓ attached to a mass m. The pendulum is hung in a tower that is at latitude λ on the earth's surface[2], so it is natural to describe its motion with coordinates fixed to the rotating Earth. Let ω (i.e. once per day) be the Earth's angular velocity. Use either the (x, y, z) or (r, θ, ϕ) coordinates shown in the figure. Here z is perpendicular to the Earth's surface and y is tangent to a circle of constant longitude that passes through the north pole, and x therefore points east. The radius of the earth is R_e



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[2] 0^o latitude is the equator, 90^o latitude is the north pole

(b) Determine the Lagrangian of the Pendulum. From the start you may keep terms up to first order in ω, and of course you may neglect total time derivatives to simplify the analysis. Derive the Lagrangian for the pendulum small oscillations. I find

$$L = \frac{1}{2}m\ell^2\left[(\dot{\theta})^2 + \theta^2\dot{\phi}^2\right] - m\omega\ell^2\dot{\phi}\sin(\lambda)\theta^2 \; - mg\ell\frac{\theta^2}{2} \tag{9}$$

though in retrospect it may have been easier to use the xy coordinate system.

(c) Demonstrate that the pendulum undergoes precession with a rate $\dot{\phi} = \omega\sin\lambda$, by exactly solving the equations of motion for the small oscillations. Hint: it may be helpful to change variables back to Cartesian coordinates

$$x \equiv \ell\theta\sin(\phi) \tag{10}$$

$$y \equiv \ell\theta\cos(\phi) \tag{11}$$

before determining the equations of motion. The resulting equations can be solved exactly, by introducing $z(t) = x + iy$, and solving for z. Then the x and y coordinates may be recovered by taking the real and imaginary parts. Describe carefully which way the pendulum precesses.

Problem 3. Preliminaries

Answer as briefly as possible!

(a) Given a tensor $\boldsymbol{I} = I_{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ in the rotating basis and in the fixed basis[3] $\boldsymbol{I} = \underline{I}_{ab}\underline{\boldsymbol{e}}_a \otimes \underline{\boldsymbol{e}}_b$ (here $\boldsymbol{e}_a = R_{ab}\underline{\boldsymbol{e}}_b$), show that the components are related via

$$I_{ab} = R_{ac}\, R_{bd}\, \underline{I}_{cd}\,. \tag{12}$$

Express this transformation rule with matrices.

(b) Show that

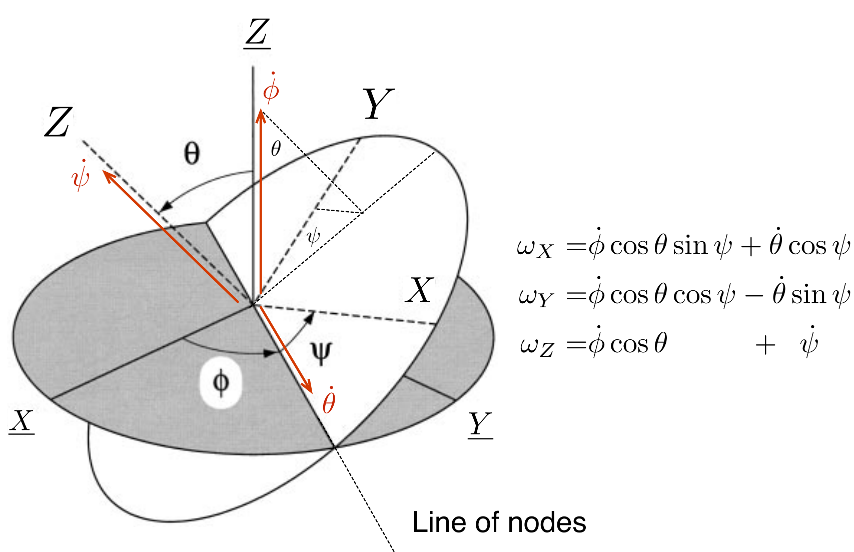
$$\boldsymbol{w} \times \boldsymbol{v} = \hat{\boldsymbol{v}} \cdot \boldsymbol{w} = \boldsymbol{v} \cdot \hat{\boldsymbol{w}} \tag{13}$$

where (for example) $\hat{\boldsymbol{v}} = \hat{v}_{ab}\ \boldsymbol{e}_a \otimes \boldsymbol{e}_b$ denotes the antisymmetric tensor $\hat{v}_{ab} = \epsilon_{abc}v^c$ associated with the vector $\boldsymbol{v}$. Express these two alternate forms of the cross product using matrices.

(c) Show that $\underline{\omega}_{ac} = (R^{-1}\dot{R})_{ac}$

(d) Determine the projection of $\vec{\omega}$ on to the lab frame axes $\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3$. (You may use either algebraic means, computer algebraic means, or use the appropriate picture from lecture, or all three means.) You should find

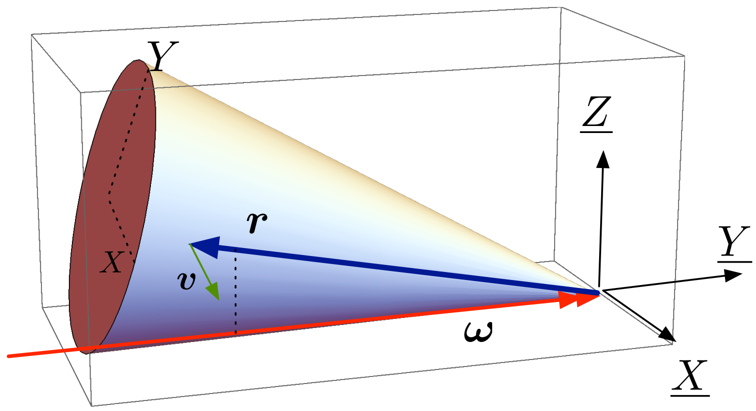
$$\begin{pmatrix} \omega_{\underline{X}} \\ \omega_{\underline{Y}} \\ \omega_{\underline{Z}} \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\phi) \\ \dot{\theta}\sin(\phi) - \dot{\psi}\sin(\theta)\cos(\phi) \\ \dot{\psi}\cos(\theta) + \dot{\phi} \end{pmatrix} \tag{14}$$



[3] Often I will write $\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ as simply $\boldsymbol{e}_a\boldsymbol{e}_b$ with the $\otimes$ implied. Then $\boldsymbol{I} \cdot \boldsymbol{v}$ takes the dot product with the second slot $\boldsymbol{I} \cdot \boldsymbol{v} = I_{ab}v^b\, \boldsymbol{e}_a$, while $\boldsymbol{v} \cdot \boldsymbol{I}$ takes the dot product with the first, $v^a I_{ab}\boldsymbol{e}_b$.

Problem 4. A Rolling Cone (Adapted from Goldstein Ch.5 #17)

A uniform right circular cone of height h, half-angle α, and density ρ rolls on its side without slipping on a uniform horizontal plane. It returns to its original position in a time τ .



(a) Find the moment of inertia tensor for the body (or principal) axes centered on the tip. I find

$$I^0 = \frac{3}{5}Mh^2 \begin{pmatrix} \frac{1}{4}\tan^2\alpha + 1 & & \\ & \frac{1}{4}\tan^2\alpha + 1 & \\ & & \frac{1}{2}\tan^2\alpha \end{pmatrix} \tag{15}$$

(b) The cone is turning around the $\underline{Z}$ axis in a counterclockwise fashion as seen from above. Consider the infinitesimal rotation at $t = 0$ (see figure) that the cone experiences – the displacement of a point $\boldsymbol{r}$ on the cone's body is

$$\boldsymbol{r} \to \boldsymbol{r} + \delta\boldsymbol{\theta} \times \boldsymbol{r}\,, \tag{16}$$

where $\delta\boldsymbol{\theta}$ points along the $\underline{Y}$ axis. Describe qualitatively why Eq. (16) (with the specified direction of $\boldsymbol{\omega}$) is what we mean by a rolling cone. Argue in particular that $\underline{\omega}_z = 0$ and write down the components of $\boldsymbol{\omega}(t)$ in the lab frame.

(c) Determine the Euler angles describing the cone as a function of time. Take the Z axis to point along the axle of the cone. Interpret $\dot\phi$ and the relation between $\dot\psi$ and $\dot\phi$.

(d) Find the kinetic energy of the rolling cone. I find

$$T = Mh^2\left(\frac{2\pi}{\tau}\right)^2\left[\frac{3}{40}(1 + 5\cos^2\alpha)\right] \tag{17}$$

(e) (Optional.) Write down the components of the $\boldsymbol{L}(t)$ in the lab frame. (You may wish to check your results by computing $T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$)

(f) (Optional.) There are two ways to compute the kinetic energy. The first way uses the expression

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot I_{\rm tip}\cdot\boldsymbol{\omega}\,. \tag{18}$$

where I_{tip} is the moment of inertia around the tip. The second way uses the moment of inertia of the center of masss I_{cm}

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot I_{\text{cm}} \cdot \boldsymbol{\omega} + \frac{1}{2}M\boldsymbol{v}_{\text{cm}}^2 \,. \tag{19}$$

Show that these are equivalent to each other provided I_{cm} and I_{tip} are related by the parallel axis theorem.