Problem 1. Nutation of a Heavy Symmetric Top

Consider a heavy symmetric top with one end point fixed.

(a) Write down the Lagrangian from class. Carry out Routh's procedure explicity by Legendre transforming with respect to the the conserved momenta p_ψ and p_ϕ. Write down $-R$ which serves as effective Lagrangian $L_{\rm eff}$ for θ. Show that θ obeys the equation of motion following from this effective Lagrangiang

$$I\ddot{\theta} = -\frac{\partial U_{\rm eff}}{\partial \theta}\,, \tag{1}$$

where

$$U_{\rm eff} = mg\ell\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta}\,. \tag{2}$$

Also show that

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2(\theta)}\,. \tag{3}$$

(b) In class we analyzed the limit when gravitational torque is small to the rotational kinetic energy, $mg\ell/(p_\psi^2/I_1) \ll 1$. Take $p_\phi/p_\psi = r$ with $0 < r < 1$. Within this approximation (known as the fast top approximation), if the energy E is adjusted to the minimum of the effective potential, the tip of the top will slowly precess with

$$\dot{\theta} = 0\,, \quad \text{and} \quad \dot{\phi} = \frac{mg\ell}{p_\psi}\,. \tag{4}$$

This is is shown in Fig. 1(d) which shows the trajectory of the tip of the top on the sphere.

Now if the energy of the system is slightly larger than the minimum of $U_{\rm eff}$, describe qualitatively the motion in θ and ϕ. For what range in E do the first (a) and second (b) figures describe the top's motion? Explain. Work in the fast top approximation

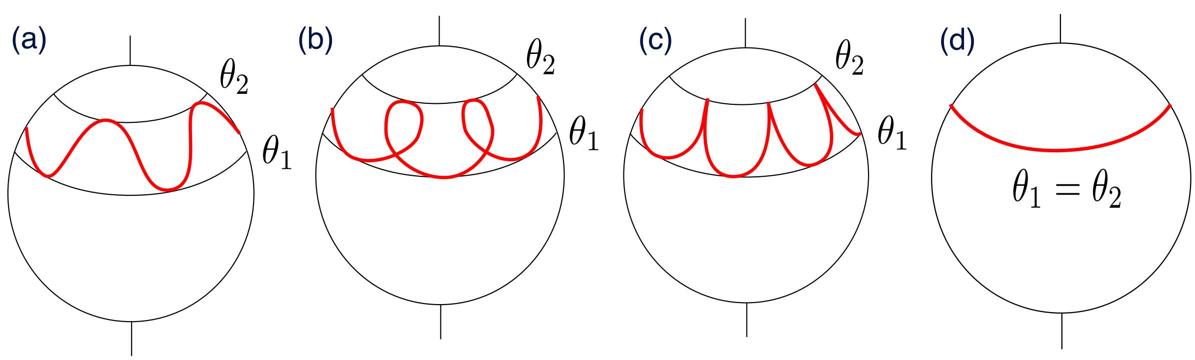
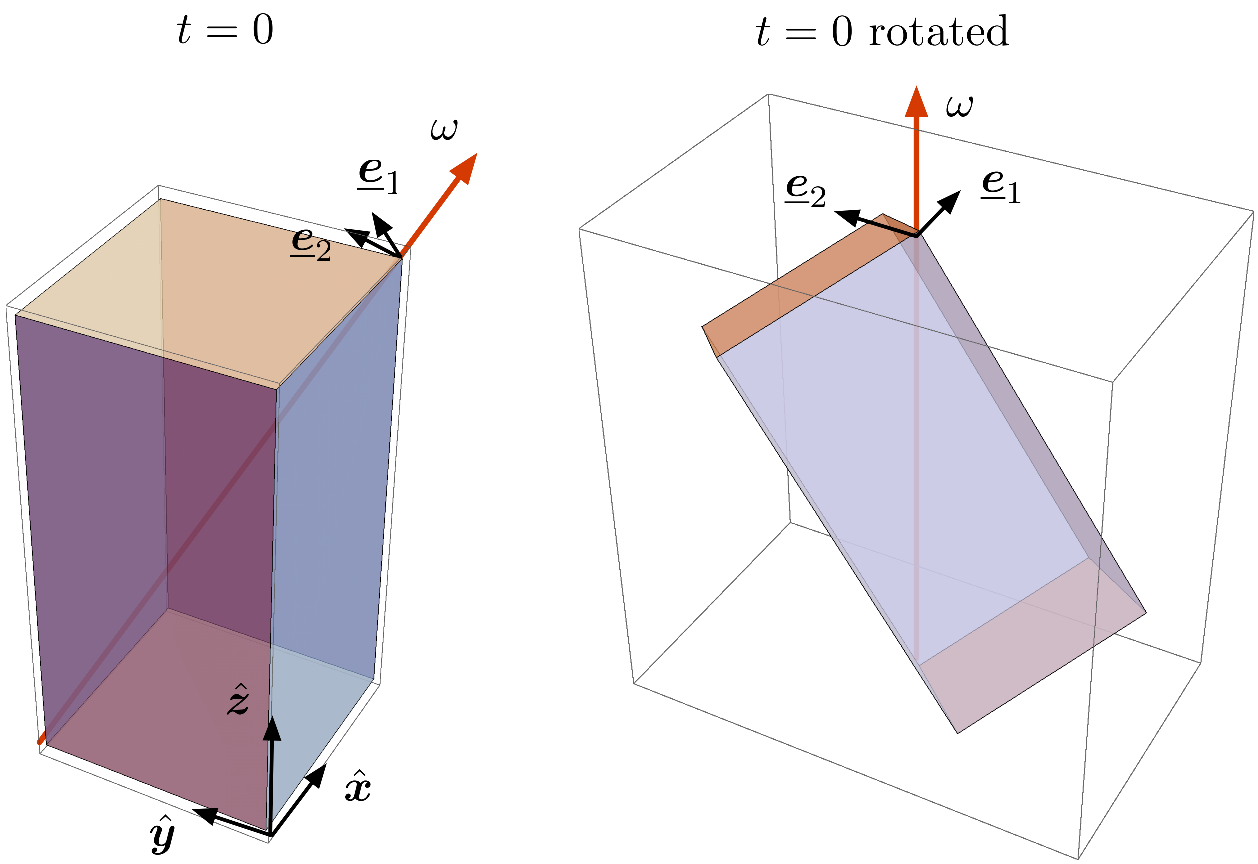


Figure 1: Motion of the tip of the heavy symmetric top

(c) Using the fast top approximation outlined in (b), compute the period of oscilations for a given energy E, and determine the precession rate $\dot{\phi}(t)$, and angle $\theta(t)$, as a function of time. What is the average precession rate?

Problem 2. Torque on a box

Consider a solid box of mass m and dimension $L, L, 2L$ (see figure).



(a) Compute all components of the moment of inertia tensor around center of mass.

(b) The box is rotated with constant angular frequency ω around its diagonal. At $t = 0$ the box is oriented so that its principal axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are aligned with laboratory $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ as shown in the figure. Compute the components of angular momentum as a function of time in the body basis and in the lab basis. For the lab basis you use the fixed basis vectors $\underline{\boldsymbol{e}}_1$, $\underline{\boldsymbol{e}}_2$, $\underline{\boldsymbol{e}}_3$ shown in the figure, which differ by a *constant* rotation from $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.

$$\underline{\boldsymbol{e}}_1 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) \tag{5}$$

$$\underline{\boldsymbol{e}}_3 = \frac{1}{\sqrt{6}}(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}) \tag{6}$$

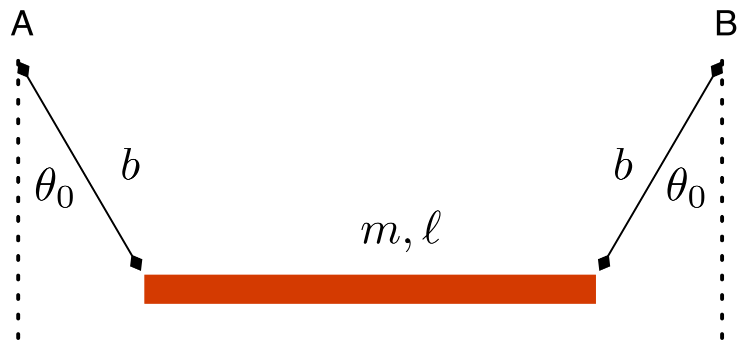
$$\underline{\boldsymbol{e}}_2 = \underline{\boldsymbol{e}}_3 \times \underline{\boldsymbol{e}}_1 \tag{7}$$

(c) Compute the components of the torque required to maintain the box's rotational motion working with the rotating basis. Compute the components of the torque working with the fixed basis.

(d) (Optional) Use the Lagrangian framework to compute the required torques in the body frame.

Problem 3. Oscillations of a bar

A uniform bar of length ℓ and mass m lies in the xy plane and is attatched by two equal springs of equilibrium length b and force constant k as shown in the figure below. (Gravity is to be ignored, it points in the z direction). The points A and B are held fixed (and are separated by a distance $\ell + 2b\sin\theta_0$ see below), but the bar is otherwise free to translate and rotate in the xy plane. Determine the normal modes of small oscillations in the plane and the associated frequencies.



You should find the oscillation frequencies

$$\frac{\omega^2}{k/m} = 0, 2\cos\theta_0^2, 4 + 2\cos(2\theta_0) \qquad (8)$$

Give an interpretation of the zero mode.