Kepler's Laws

- Motion of earth & sun

$$L = \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

$$\xleftrightarrow{\quad KE \quad}$$

- Total momentum is constant. Define

$$\left.\begin{aligned} M_{TOT} \vec{R} &= m_1 \vec{r}_1 + m_2 \vec{r}_2 \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}\right\} \qquad \begin{aligned} \vec{r}_1 &= \vec{R} + \frac{m_2}{M_{TOT}} \vec{r} \\ \vec{r}_2 &= \vec{R} - \frac{m_1}{M_{TOT}} \vec{r} \end{aligned}$$

- Then Lagrangian reads

$$L = \frac{1}{2} M_{TOT} \dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2} \mu \dot{\vec{r}}^{\,2} - U(r) \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

- Now $\vec{L} = \vec{r} \times \vec{p}$ is constant. So, we can choose $\vec{r}$ to lie in the x, y plane, wich is perpendicular to $\vec{L}$ which can be taken to be directed along the z-axis

So

$$\vec{r} = (r\cos\phi, r\sin\phi, 0)$$

And so neglecting the CM motion, the lagrangian is

$$\boxed{L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)}$$

The EOM

means L is independent of ϕ and is cyclic

- $p_\phi = \dfrac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}$, and

$$\frac{d}{dt}(\mu r^2 \dot{\phi}) = 0 \implies \boxed{p_\phi = \text{constant} \equiv \ell}$$

- $$\frac{d}{dt}(\mu \dot{r}) = \mu r \dot{\phi}^2 - \frac{\partial U(r)}{\partial r}$$

$$= \frac{p_\phi^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

use definition of p_ϕ

$$= -\frac{\partial}{\partial r}\Big(\underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{\equiv V_{eff}(r)}\Big)$$

Perhaps, we should write $V_{\text{eff}}(r, p_\phi)$ to emmphasize that p_ϕ (as opposed to $\dot{\phi}$) is held fixed when differentiating with respect to r, i.e. the lhs is $-(\partial V_{\text{eff}}/\partial r)_{p_\phi}$

Note that one can not find $V_{eff}(r)$ by replacing $\dot{\phi}$ with p_ϕ in L. The reason is that p_ϕ is not held fixed when the Lagrangian is extremized

The EOM in the Hamiltonian setup

Given the Lagrangian

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 - U(r)$$

- $p_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$ $\qquad$ $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}$

Then

$$H = p_r \dot{r} + p_\phi \dot{\phi} - L, \quad \text{with} \quad \dot{r} = \frac{p_r}{\mu}, \; \dot{\phi} = \frac{p_\phi}{\mu r^2}$$

This yields

$$H = \frac{p_r^2}{2\mu} + \underbrace{\frac{p_\phi^2}{2\mu r^2} + U(r)}_{}$$

This is $\rightarrow V_{eff}(r, p_\phi)$

The EOM are

$$\dot{r} = \partial H / \partial p_r = \frac{p_r}{\mu}, \quad \text{and} \quad \dot{\phi} = \frac{p_\phi}{\mu r^2}.$$

- $\frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = -\frac{\partial V_{eff}}{\partial r}(r, p_\phi)$

- $\frac{dp_\phi}{dt} = -\frac{\partial H}{\partial \phi} = 0 \implies p_\phi = \text{constant}$

= angular momentum is constant

Integrating the equation of Motion

- Finally this problem has a first integral

$$E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + U(r)$$

$$= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\left(\frac{\ell^2}{2\mu r^2} + U(r)\right)}_{\equiv V_{eff}(r)}$$

Alternatively one could work with the Hamiltonian,

$$H = \frac{p_r^2}{2\mu} + \left(\frac{p_\phi^2}{2\mu r^2} + U(r)\right)$$

and $H = E$

- So the problem is essentially 1D. Solving for $\dot{r}$

$$\frac{dr}{dt} = \sqrt{\frac{2(E - V_{eff}(r))}{\mu}} \Longrightarrow dt = \sqrt{\frac{\mu}{2}}\frac{dr}{\sqrt{E - V_{eff}(r)}}$$

We find, integrating from t_0, r_0 to t, r

$$t - t_0 = \sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{E - V_{eff}(r)}}$$

- Once r is known we determine ϕ, since:

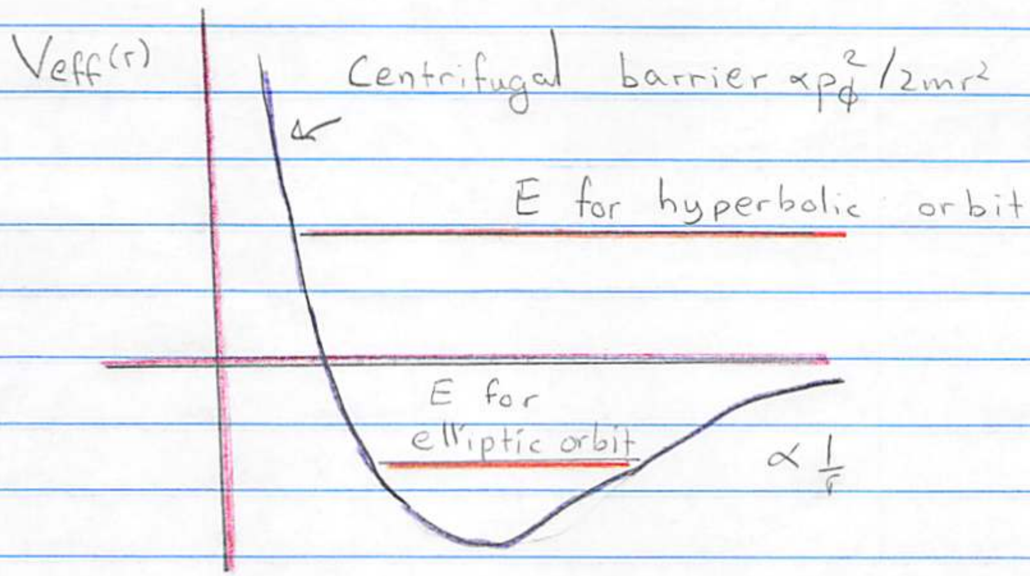
$$\frac{d\phi}{dt} = \frac{\ell}{\mu r^2} \Longrightarrow d\phi = \frac{\ell}{\mu r^2}\,dt$$

Let us express ϕ versus r

Yielding the trajectory r as a fcn of ϕ

$$\boxed{\phi - \phi_0 = \frac{\ell}{\sqrt{2\mu}} \int^{r} \frac{dr/r^2}{(E - V_{eff}(r))^{1/2}}}$$

Picture $\quad U(r) = -K/r \qquad V_{eff}(r) = U(r) + \frac{\ell^2}{2mr^2}$



- Of course we haven't shown yet that the open orbits are hyperbolas and the closed orbits are hyperbolas. We will do this shortly

N.B. The Langrangian vs Hamiltonian

- given the Lagrangian

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 - U(r) \qquad p_\phi = \mu r^2 \dot{\phi}$$

it would be a mistake to just substitute p_ϕ into L and use it to determine the EOM of r. You have to find the EOM first, and then substitute $\dot{\phi} = p_\phi / \mu r^2$

- The "problem" is that p_ϕ is not held fixed, but rather $\dot{\phi}$ is held fixed, when finding the EOM in the Lagrangian setup.

- Of course one could use the Hamiltonian (or Routhian -- see homework) where p_ϕ is fixed

$$H = \frac{p_r^2}{2\mu} + \underbrace{\left(\frac{p_\phi^2}{2\mu r^2} + U(r)\right)}_{= V_{eff}}$$

- Then the effecitive potential comes:

$$\frac{dr}{dt} = \frac{p_r}{\mu} \qquad \frac{dp_r}{dt} = -\frac{\partial V_{eff}}{\partial r}$$