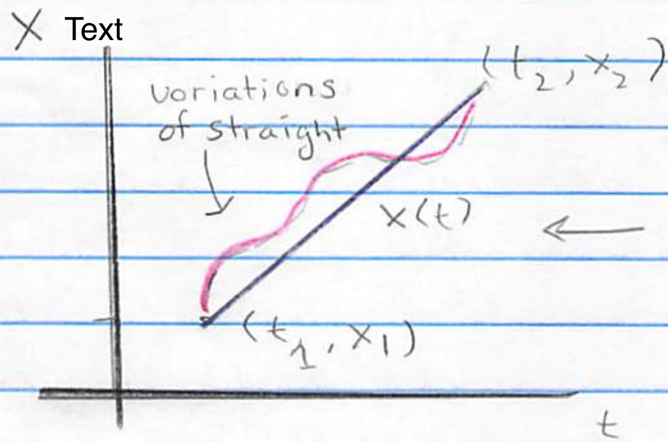
Lagrangian Formulation

- Take a free particle (no forces)

$$\frac{d}{dt}\left(m \frac{d\vec{x}}{dt}\right) = \cancelto{0}{-\frac{\partial V}{\partial \vec{x}}}$$

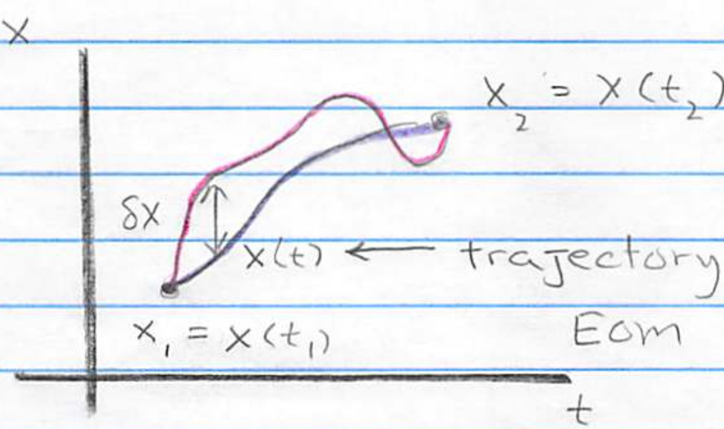


straight line trajectory in (t,x) plane.

- Clearly, at least for the free particle, this trajectory minimizes the "distance" in the (t,x) plane. This motivates us to try to formulate the whole forced motion as a minimization procedure.

called action

We will Choose the measure of "distance" to reproduce Newton.



trajectory satisfying the EOM

As for the free case we will Change (or vary) the path

☆ $\bar{x}(t) \longrightarrow \bar{x}(t) + \delta x(t)$

While keeping the endpoints fixed

☆ $\delta x(t_1) = \delta x(t_2) = 0$ (End points fixed)

- Then

$$\underbrace{S[x(t)]}_{\text{"distance" in configuration space is known as action}} = \int_{t_1}^{t_2} dt \, L(x, \dot{x}, t)$$

local measure of action: the Lagrangian

"distance" in configuration space is known as action. Takes a path and gives a number.

- Changing the path does not change the action if $x(t)$ satisfies the equation of motion

$$S[x + \delta x] = S[x] \quad \longleftarrow \quad x \text{ - must satisfy the EOM.}$$

(for this to be true)

- Now

$$S[\vec{x} + \delta\vec{x}] = \int dt \, L(\vec{x} + \delta\vec{x}, \dot{\vec{x}} + \frac{d}{dt}\delta\vec{x}, t)$$

Or

$$S[x + \delta x] = \underbrace{\int dt \, L(x, \dot{x}, t)}_{= S[x]} + \int dt \, \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{d\,\delta x}{dt}$$

- So

$$\delta S = S[x+\delta x] - S[x]$$

$$= \int dt\ \frac{\partial L}{\partial x}\,\delta x + \underline{\frac{\partial L}{\partial \dot{x}}\,\frac{d\,\delta x}{dt}}$$

integrate by parts

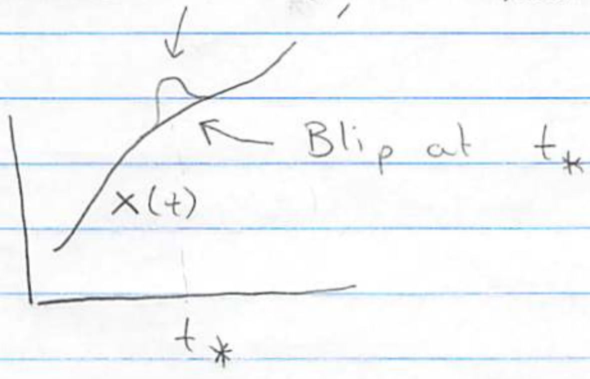
- $$\delta S = \frac{\partial L}{\partial \dot{x}}\,\delta x\,\Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x(t)$$

- For a general variation this doesn't vanish (and will be important later). But, here we required $\delta x(t_2) = \delta x(t_1) = 0$, and thus:

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x(t)$$

- We want $\delta S = 0$, and $\delta x(t)$ is arbitrary;

take $\delta x(t)$, for example, to be a "blip" at time t_*



So

$$\delta S \simeq \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\Bigg|_{t=t_*} \int dt\, \delta x(t_*) = 0$$

So require

$$\boxed{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) = 0}$$

We see that if

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

then the EOM work (i.e. agree @ Newton)

$$\frac{d}{dt}\underbrace{(m\dot{x})}_{\frac{\partial L}{\partial \dot{x}} = p} = \underbrace{-\frac{\partial V}{\partial x}}_{\frac{\partial L}{\partial x}}$$

- We have treated one particle in one dimension, but if more particles and dimensions cause no difficulty in practice

In general:

$$L = \sum_a \frac{1}{2} m_a \dot{\vec{x}}_a^{\,2} - V(\vec{x}_a)$$

And the EOM are

$$\boxed{-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{ai}} + \frac{\partial L}{\partial x_{ai}} = 0}$$

$$\begin{array}{ccc} (x, & y, & z) \\ \swarrow & \searrow & \downarrow \end{array}$$

$$\vec{x}_a = (x_{a1}, x_{a2}, x_{a3})$$

components of a-th vector,

Or

$$\boxed{-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}_a}\right) + \frac{\partial L}{\partial \vec{x}_a} = 0}$$

when clear from context