Non-linear Oscillations / Naive Perturbation Theory

(Landau 28)

- Naive perturbation theory doesn't work! Lets see what goes wrong.

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega_0^2 x^2 - \frac{1}{4} m \beta x^4$$

small non linear term.

- Then the EOM is the Duffing Equation:

$$\ddot{x} + \omega_0^2 x = -\beta x^3 \equiv \frac{f_{ind}}{m}$$ = force induced by the non-linearities

- If the amplitude, a, is small then the induced force is small compared to the harmonic force

$$\rightarrow \beta a^3 \ll \omega_0^2 a \quad \text{or} \quad \boxed{\frac{\beta a^2}{\omega_0^2} \ll 1}$$

- Now we set up a naive! perturbative expansion

$$x(t) = x^{(0)}(t) + x^{(1)}(t) + \ldots$$

Where $A \equiv a e^{i\varphi}$

$$x^{(0)} = \mathrm{Re}\left[A e^{-i\omega_0 t}\right] = a \cos(-\omega_0 t + \varphi)$$

is the zeroth order solution

- So

$$(x^{(0)})^3 = \left(\frac{A e^{-i\omega_0 t} + A^* e^{i\omega_0 t}}{2}\right)^3$$

$$= \frac{A^3}{8} e^{-3i\omega_0 t} + \frac{3}{8} A^2 A^* e^{-i\omega_0 t} + \frac{3}{8} A^{*2} A e^{i\omega_0 t} + \frac{1}{8}(A^*)^3 e^{i3\omega_0 t}$$

$\longleftrightarrow$ complex conjugate of first line

- And thus $f_{ind} \propto x^3$

$$\frac{f_{ind}}{m} = \underbrace{-\frac{\beta}{4} a^3 \cos(3(-\omega_0 t + \varphi))}_{f^{(1)} \equiv 3\omega_0 \text{ terms}} \underbrace{- \frac{3}{4}\beta a^3 \cos(-\omega_0 t + \varphi)}_{f^{(2)} \equiv \text{resonant } \omega_0 \text{ term}}$$

- So the correction $x^{(1)}$ satisfies

$$\frac{d^2 x^{(1)}}{dt^2} + \omega_0^2 x^{(1)} = \frac{f_{ind}}{m}$$

So we can use our knowledge of the harmonic oscillator to write down the solution

$$x^{(1)} = x^{(1)}_{3\omega_0} + x^{(1)}_{\omega_0} \qquad \text{with}$$

- So then the $3\omega_0$ term gives

$$x^{(1)}_{3\omega} = \frac{f^{(1)}/m}{\underbrace{-\omega^2+\omega_0^2}_{\omega=3\omega_0}} = -\underbrace{\left(\frac{\beta a^2}{\omega_0^2}\right)}_{\text{small}}\left(\frac{a}{32}\right)\cos\underbrace{(-3(\omega_0 t+\varphi))}_{\text{higher harmonics}}$$

- So we see how non-linearities generate higher harmonics $\propto (e^{-i\omega t})^3$.

- The resonant terms give

$$x^{(1)}_{\omega_0} = \frac{f^{(2)}}{2m\omega_0^2}\,\omega_0 t\sin\omega_0 t = -\frac{3}{8}\underbrace{\left(\frac{\beta a^2}{\omega_0^2}\right)}_{\text{small}}\underbrace{\omega_0 t}_{\text{but this gets large!}}\sin\omega_0 t$$

secular term

- These resonant terms grow with time, destroying the perturbation theory

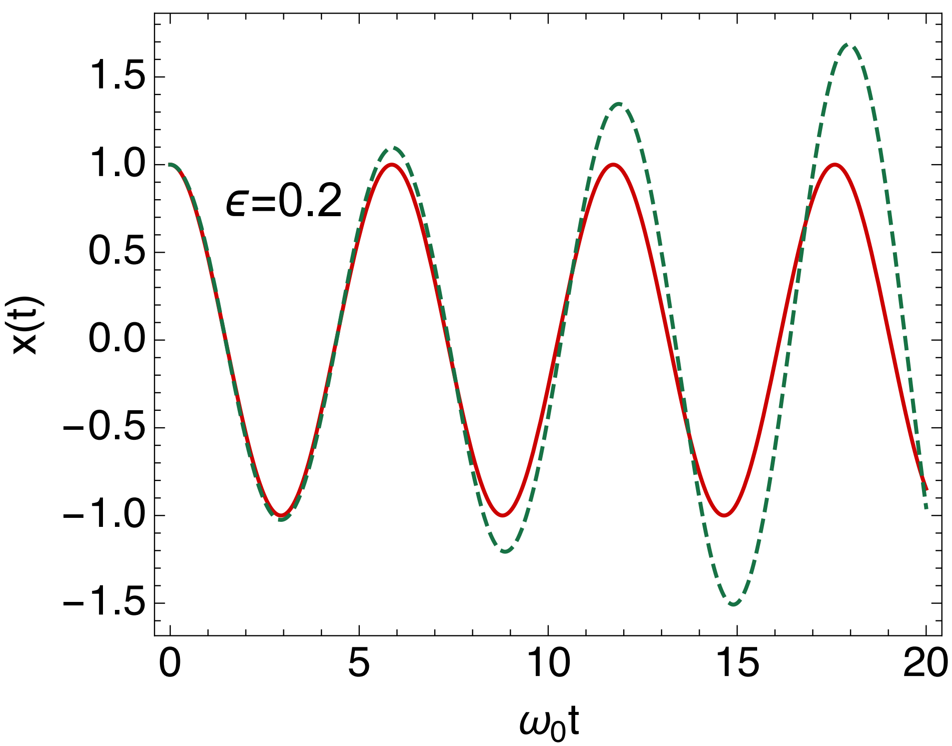
The figure on the next page shows

$$x = x^{(0)} + x^{(1)}_{3\omega_0} + x^{(1)}_{\omega_0}$$

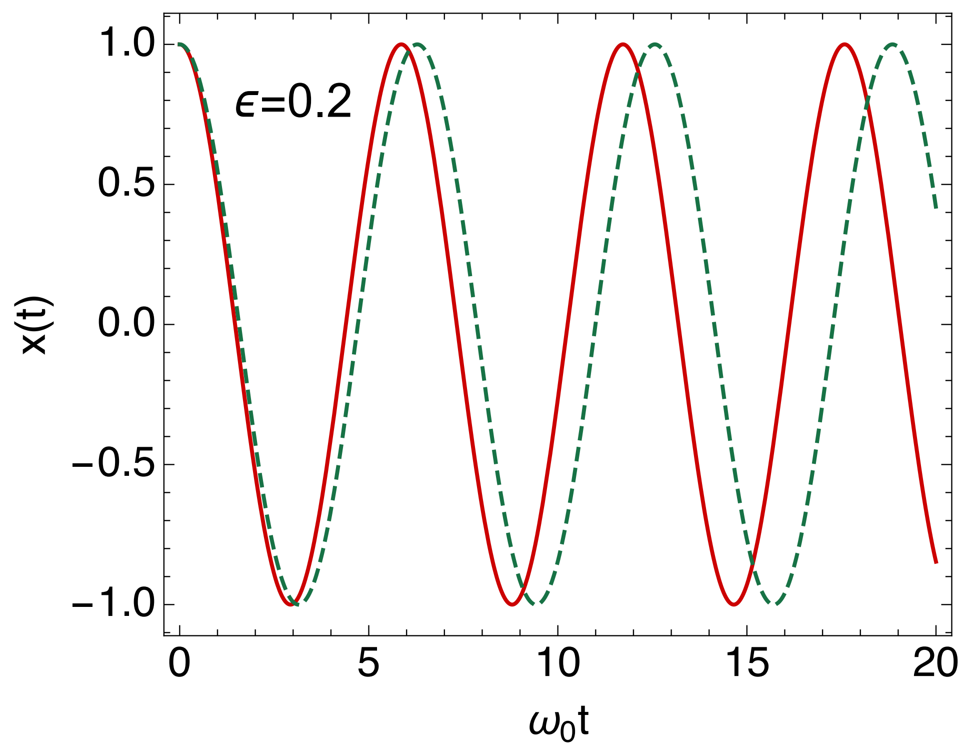
with and without the secular term $\propto t\sin\omega_0 t$

Solid lines exact solution, dashed lines approximate

Non linear oscillator
with secular terms



Non linear oscillator
w/out secular terms



$$\epsilon \equiv \frac{\beta a^2}{\omega_0^2}$$

- The problem is that although $\frac{\beta a^2}{\omega_0^2} \ll 1$, for late times $\frac{\beta a^2}{\omega_0}(\omega_0 t) \gg 1$.
  secular divergence

- To have a useful perturbative expansion one must resum the secular divergences;

Examining the figure,

- We need to shift the frequency:

$$\cos((\omega_0 + \Delta\omega)t) = \cos\omega_0 t \cos\Delta\omega t - \sin\Delta\omega t \sin\omega_0 t$$

$$\simeq \cos\omega_0 t - \Delta\omega t \sin\omega_0 t$$

Comparison with ☆ on the previous page gives

$$\Delta\omega = \frac{3}{8}\left(\frac{\beta a^2}{\omega_0^2}\right)\omega_0$$

(Justified in the next section)

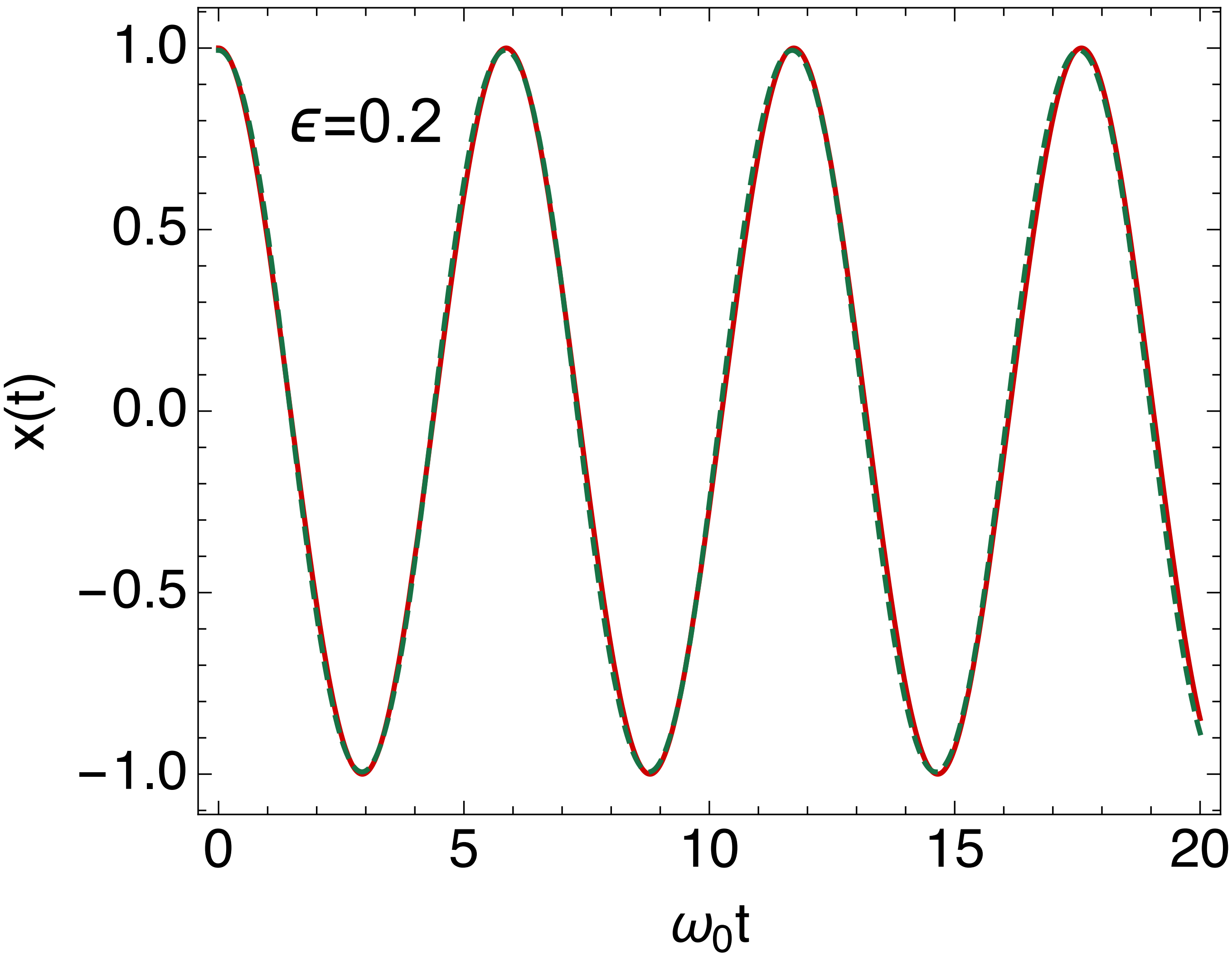
- We will justify this formally in the next section. Defining $\varepsilon \equiv \beta a^2/\omega_0^2 \ll 1$, our approximate solution (see next page) is

$$\boxed{x(t) = a\cos\left(\omega_0\left(1 + \frac{3}{8}\varepsilon\right)t\right) - \frac{\varepsilon}{32} a\cos(3\omega_0 t)}$$

And this reproduce the full solution wonderfully!

Non linear oscillator treating secular term as frequency shift

$$\epsilon = \frac{\beta a^2}{\omega_0^2}$$



Solid lines exact solution, dashed lines approximate