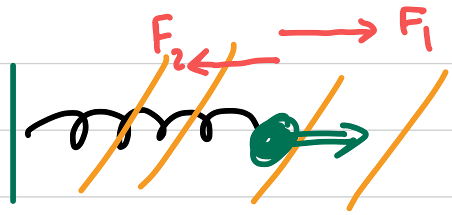
Brownian Motion of SHO



The damped SHO sits in a medium at temperature T. The mass is large so that it moves for a long time compared to the time scales of the medium. Thus we can treat the forces provided by the medium as random

$$m \frac{d^2x}{dt^2} + m\eta \frac{dx}{dt} + m\omega_0^2 x = F(t)$$

random force or kicks

☆ $F_1 \equiv F(t_1)$ is in a random direction compared to $F_2 \equiv F(t_2)$. Both of these forces last only a short time (see picture)

This means

Amplitude of force

$$\langle F(t) \rangle = 0$$

$$\langle F(t) F(t') \rangle = A\, \delta(t-t')$$

As we will show, the oscillator will not equilibrate unless the random force has $A = 2Tm\eta$. This is because the drag slows you down, while the kicks speed you up. To reach equilibrium these need to balance.

- Some elementary stat. mech gives the average energy of the oscillator after a long time (not part of course)

The probability to have position x and momentum p is

$$P(x,p) = C\, e^{-H/T} \qquad H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x$$

So

$$\bar{E} = \frac{\int dx\,dp\, P(x,p)\, H}{\int dx\,dp\, P(x,p)} = T$$

all we need is $\bar{E} = T$

- Now return to mechanics where if the partice starts at $x=0$ and $v=0$ at $t=0$, after a time t we find

$$x = \int_0^\infty G_R(t,t_0)\, F(t_0)\, dt_0$$

☆ $$x(t) = \int_0^t dt_0 \frac{e^{-\eta/2\,(t-t_0)} \sin(\omega_0(t-t_0))}{m\omega_0} F(t_0)$$

Note

☆☆ $$\dot{x}(t) \simeq \int_0^t dt_0 \frac{e^{-\eta(t-t_0)/2} \cos(\omega_0(t-t_0))}{m\omega_0}\, \omega_0 + \underbrace{O(\eta)}_{\text{damping is small}}$$

- Now lets compute the energy of the SHO versus time

$$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2$$

Substitute $\star$ and $\star\star$

$$\overline{E(t)} = \int_0^t dt_0 \int_0^t dt_0' \; \frac{\overline{F(t_0) F(t_0')}}{2m} \, e^{-\frac{\eta}{2}(t-t_0)} e^{-\frac{\eta}{2}(t-t_0')}$$
$$\times \Big[\cos(\omega_0(t-t_0)) \cos(\omega_0(t-t_0')) + \sin \omega_0 (t-t_0) \sin \omega_0 (t-t_0') \Big]$$

We now compute the average using

$$\overline{F(t_0) F(t_0')} = A \, \delta(t_0 - t_0')$$

This sets $t_0' = t_0$ and collapses the t_0' integral. Using $\cos^2 + \sin^2 = 1$, we find

$$\overline{E}(t) = \int_0^t dt_0 \frac{A}{2m} e^{-\eta(t-t_0)}$$

$$= \frac{A}{2m\eta} \left(1 - e^{-\eta t}\right) \xrightarrow[t \to \infty]{} \frac{A}{2m\eta}$$

- Now at late times we must find $\bar{E} = T$. This leads to the so called fluctuation dissipation relation $A = 2Tm\eta$ or

$$\boxed{\langle F(t) F(t') \rangle = 2Tm\eta\, \delta(t-t')}$$

expressing the noise $F(t)$ in terms of the dissipative parameter.