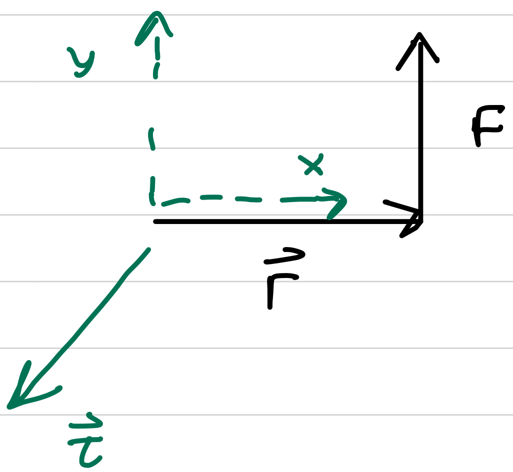
Cross Products and Antisymmetric Tensors

- Consider the torque



The torque "lives" in the x-y plane. But "points" in the z-direction. This motivates the following association

- For every vector $\vec{v} = v^a \vec{e}_a$ there is an associated anti-symmetric tensor $\hat{v}$

$$\hat{v}_{ab} \equiv \varepsilon_{abc} v^c$$

math people call $\hat{v}$ the "hodge-dual" of $\vec{v}$

and the inverse relation

$$v^a = \frac{1}{2} \varepsilon^{abc} \hat{v}_{bc}$$

i.e.

$$\hat{v} = \begin{pmatrix} 0 & \hat{v}_{xy} & \hat{v}_{xz} \\ -\hat{v}_{xy} & 0 & \hat{v}_{yz} \\ -\hat{v}_{xz} & -\hat{v}_{yz} & 0 \end{pmatrix} = \begin{pmatrix} 0 & v^z & -v^y \\ -v^z & 0 & v^x \\ v^y & -v^x & 0 \end{pmatrix}$$

So for torque $\hat{\tau}_{xy} = \tau^z$!

- Antisymmetric tensors express the cross product naturally:

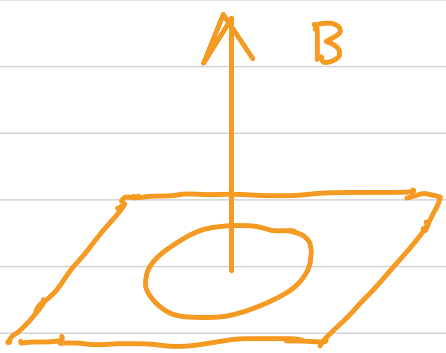
$$\vec{V} \times \vec{W} = \varepsilon_{abc} V^a W^b \vec{e}^{\,c}$$
$$= W^b \hat{V}_{bc} \vec{e}^{\,c}$$

i.e.

$$\boxed{(\vec{V} \times \vec{W})_c = W^b \hat{V}_{bc} \quad \text{or} \quad \vec{V} \times \vec{W} = \vec{W} \cdot \hat{V}}$$

← reversed order

- This is very common in physics; e.g.

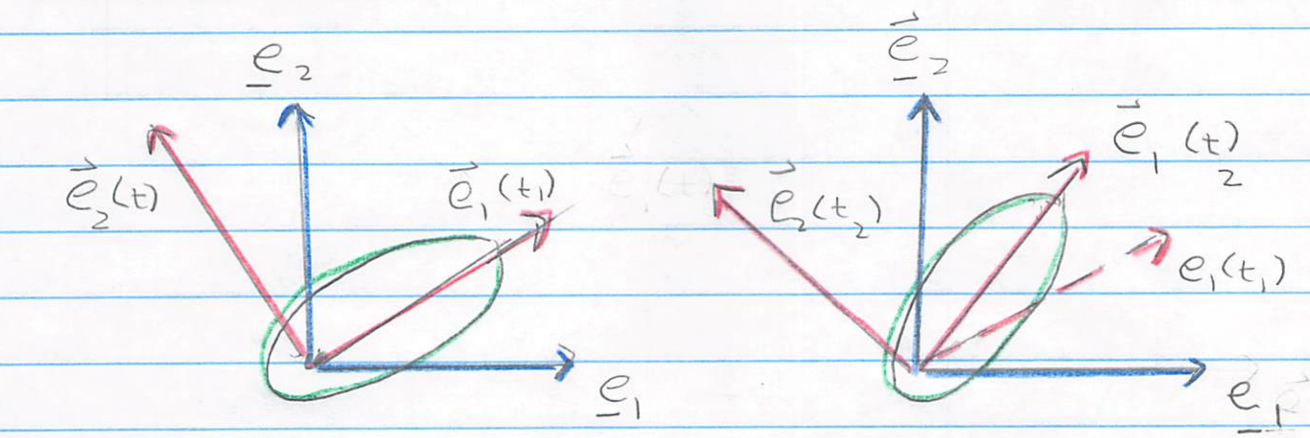


A magnetic field in the z direction "lives" in the xy plane

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B^z = \partial_x A_y - \partial_y A_x$$
$$= \hat{B}_{xy} \; !$$

Kinematics of Rotation – Follows Tong



- Then we specify a fixed orthonormal basis
  - $\underline{\vec{e}}_a \equiv$ fixed $\qquad \underline{e}_a \cdot \underline{e}_b = \delta_{ab}$
- And a time dependent orthonormal basis
  - $\vec{e}_a(t) \equiv$ time dependent basis which moves with the body

$$\vec{e}_a(t) \cdot \vec{e}_b(t) = \delta_{ab}$$

- There is a rotation matrix connecting $\{\underline{\vec{e}}_a\}$ to $\{\vec{e}_a(t)\}$

$$\star \qquad \vec{e}_a(t) = R_{ac}(t)\,\underline{\vec{e}}_c$$

Clearly (taking the dot-product of $\star$ with $\underline{\vec{e}}_b$) give

$$\boxed{\vec{e}_a(t) \cdot \underline{\vec{e}}_b = R_{ab}(t) = \text{matrix of cosines}}$$

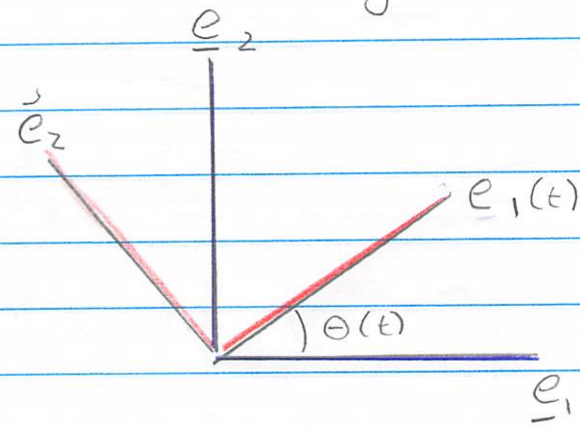
- For the rotations we are drawing:

$$R_{11} = e_1 \cdot \underline{e}_1 = \cos\theta$$

$$R_{12} = e_1 \cdot \underline{e}_2 = \sin\theta$$

$$R_{21} = e_2 \cdot \underline{e}_1 = -\sin\theta$$

$$R_{22} = \cos\theta$$



So

$$(R) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Notations:

row a

$$(R)_{ab} = R_{ab} = (R^T)_{ba}$$

entry of matix, some would use r_{ab} #'s to keep separate entries and matrices.

matrix R

column b

transpose of R

$$(RR^T)_{ab} = (R)_{ac}(R^T)_{cb} = R_{ac}R_{bc}$$

Claim: R_{ab} is an orthogonal matrix, ie, $R^{-1} = R^T$

Prf $\quad \vec{e}_a(t) \cdot \vec{e}_b(t) = \delta_{ab}$

$$R_{ac}\, \underline{e}_c \cdot R_{bd}\, \underline{\vec{e}}_d = \delta_{ab}$$

Or $\quad R_{ac} R_{bc} = \delta_{ab}$ i.e. $(R\, R^T)_{ab} = \delta_{ab}$

And thus $RR^T = \mathbb{1}$ is a orthogonal matrix

Angular velocity

- Pick a point $\vec{r}$ fixed on the rotating body

$$\vec{r} = r_a\, \vec{e}_a(t)$$

changing in time

fixed in time

- So

$$\frac{d\vec{r}}{dt} = r_a \frac{d\vec{e}_a}{dt} = r_a \dot{R}_{ab}\, \underline{\vec{e}}_b$$

insert $R^{-1}R$

$$= r_a (\dot{R}_{ab} R^{-1}_{bc})\, \vec{e}_c$$

$$\equiv r_a\, \hat{\omega}_{ac}\, \vec{e}_c \qquad \boxed{\hat{\omega}_{ac} \equiv (\dot{R}_{ab} R^{-1}_{bc})}$$

- $\hat{\omega}_{ab}$ is the angular velocity matrix

- We will show below that $\hat{\omega}$ is anti-symmetric
- We derived for $\vec{r} = r_a \vec{e}_a$:

$$\frac{d\vec{r}}{dt} = \underbrace{r^a \hat{\omega}_{ab} \vec{e}_b}_{\vec{r}\cdot\hat{\omega}}$$

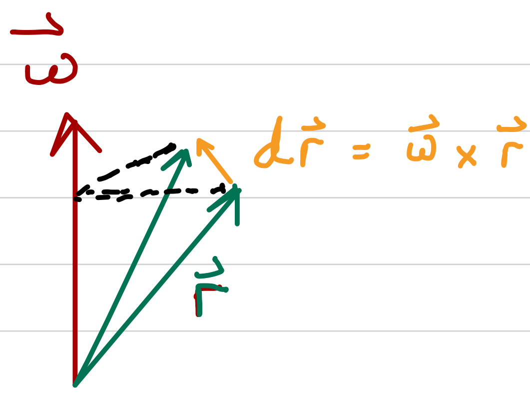
This is what we called $\vec{r}\cdot\hat{\omega}$, which is naturally expressed as a crossproduct $\vec{r}\cdot\hat{\omega} = \vec{\omega}\times\vec{r}$. So

$$\star \quad d\vec{r}/dt = \vec{\omega}\times\vec{r}$$

- Finally since $\vec{r}$ was an arbitrary fixed vector we can apply $\star$ to the basis vectors

$$\boxed{\frac{d\vec{e}_a}{dt} = \vec{\omega}\times\vec{e}_a} \quad \text{or} \quad \boxed{\frac{d\vec{e}_a}{dt} = \hat{\omega}_{ab}\vec{e}_b}$$

Physical picture:



$\omega^z = \hat{\omega}_{xy}$

rotation in xy plane

- If the components of $\vec{r}$ are not fixed, $\vec{r}(t) = r^a(t)\,\vec{e}_a(t)$, then:

$$d\vec{r}/dt = (dr^a/dt)\,\vec{e}_a + r^a \underbrace{d\vec{e}_a/dt}_{\vec{\omega}\times\vec{e}_a}$$

$$\boxed{\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\vec{r}}$$

Here $(d\vec{r}/dt)_r \equiv \frac{dr_a}{dt}\vec{e}_a$, is the derivative "in" the rotating frame.

- $\hat{\omega}$ is the generates infinitessimal rotations.

$$\frac{d\vec{e}_a}{dt} = \hat{\omega}_{ab}\,\vec{e}_b \implies \vec{e}_a(t+\Delta t) = \vec{e}_a + \hat{\omega}_{ab}\,\Delta t\,\vec{e}_b$$

So

$$\vec{e}_a(t+\Delta t) = \underbrace{(\mathbb{1} + \hat{\omega}\,\Delta t)_{ab}}\,\vec{e}_b(t)$$

Thus the matrix $\mathbb{1} + \hat{\omega}\,\Delta t$ takes the basis at t, and rotates it to $t+\Delta t$.

- **Proof that $\hat{\omega}$ is Anti Symmetric**

Since $R R^T = \mathbb{1}$ or $R_{ab} R_{cb} = \delta_{ac}$

we have

$$\dot{R}_{ab} R_{cb} + R_{ab} \dot{R}_{cb} = 0$$

$$\omega_{ac} + \omega_{ca} = 0 \implies \omega_{ac} = -\omega_{ca}$$

We used that

$$R^T = R^{-1}$$

$$\dot{R}_{ab} R_{cb} = \dot{R}_{ab} (R^T)_{bc} = (\dot{R} R^{-1})_{ac} = \omega_{ac}$$

- Aside:

$$\frac{d\vec{r}}{dt} = \left(\frac{dr^a}{dt} \delta_{ab} + r^a \hat{\omega}_{ab} \right) \vec{e}_b$$

$$\equiv (D_t r)_b \, \vec{e}_b$$

$\hat{\omega}^{(t)}_{ab}$ is the "connection"

The combination is:

$$(D_t r)_b \equiv \frac{dr^a}{dt} \delta_{ab} + r^a \hat{\omega}_{ab} \,,$$

and is called a covariant derivative!

$(D_t r)_a$ is the convariant derivative of r_a. $\hat{\omega}_{ab}$ is very analogous to the time component of a non-abelian gauge field in particle physics