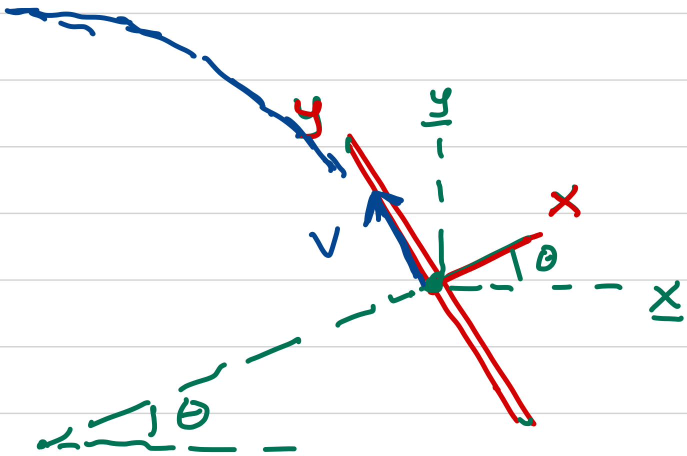
Angular Velocity Again

- Recall that $\hat{\omega}_{ab} \equiv \dot{R}_{ac} R^{-1}_{cb} = (\dot{R} R^{-1})_{ab}$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



- Consider the red object moving in a circle

- ω records the orientation of x, y relative to the $\underline{x}, \underline{y}$. Don't worry about the cm motion.

nothing more

$$\hat{\omega}_{ab} = (\dot{R} R^{-1})_{ab} = \begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

- So $\omega = \dot{\theta}$. Now we will do the same thing in 3D!

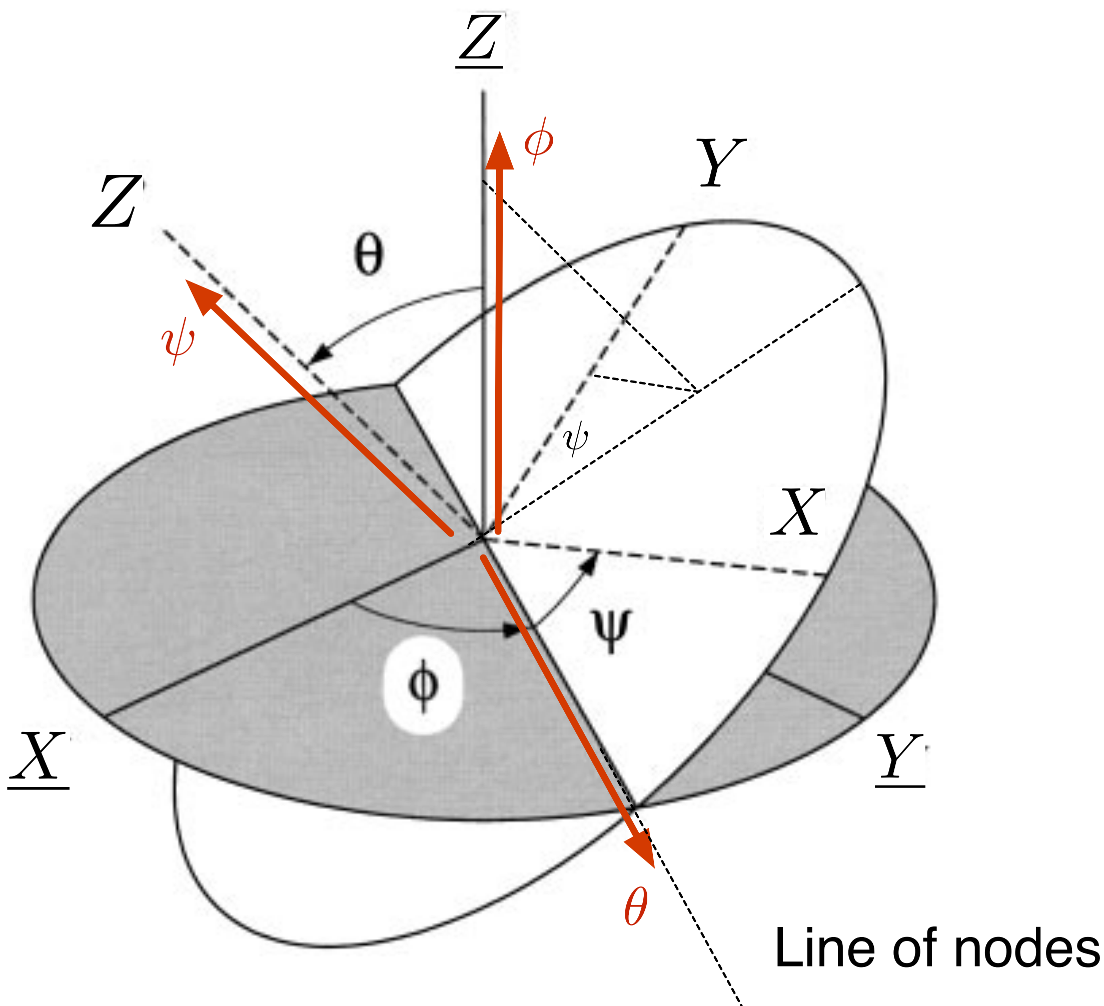
Determining The Rotation - Euler Angles

- Now we will move on and reconstruct the orientation of the body from the angular velocities
- This is analogous to finding θ from $\omega(t)$
- The rigid body is characterized by three angles, known as Euler Angles
- A sequence of rotations can take $\underline{X}, \underline{Y}, \underline{Z}$ to X, Y, Z

The steps are:

1. First a rotation by ϕ around $\underline{Z}$:
2. Then a rotation by θ around the new X axis (The line of nodes)
3. Finally a rotation around the new Z-axis by an angle ψ

- The first handout shows the steps graphically
- The second handout shows the result mathematically



- First rotate by ϕ around $\underline{Z}$
- Then rotate by θ around the line of nodes
- Finally rotate by ψ around Z

Three combined rotations

$$
R = \underbrace{\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_3(\psi)} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}}_{R_1(\theta)} \underbrace{\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_3(\phi)}
$$

$$
= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}
$$

- Now what is the relation between the $\vec{\omega}$ and the rotation matrix R

Recall that

$$\hat{\omega}_{ab} \equiv \dot{R}_{ac} R^{-1}_{cb}$$

- So in principle a few key strokes with Mathematica determines $\hat{\omega}_{ab}$, with e.g. $\hat{\omega}_{12} = \omega^3$

- This will work. However, as physicists we can use the picture on the handout to write, for instance:

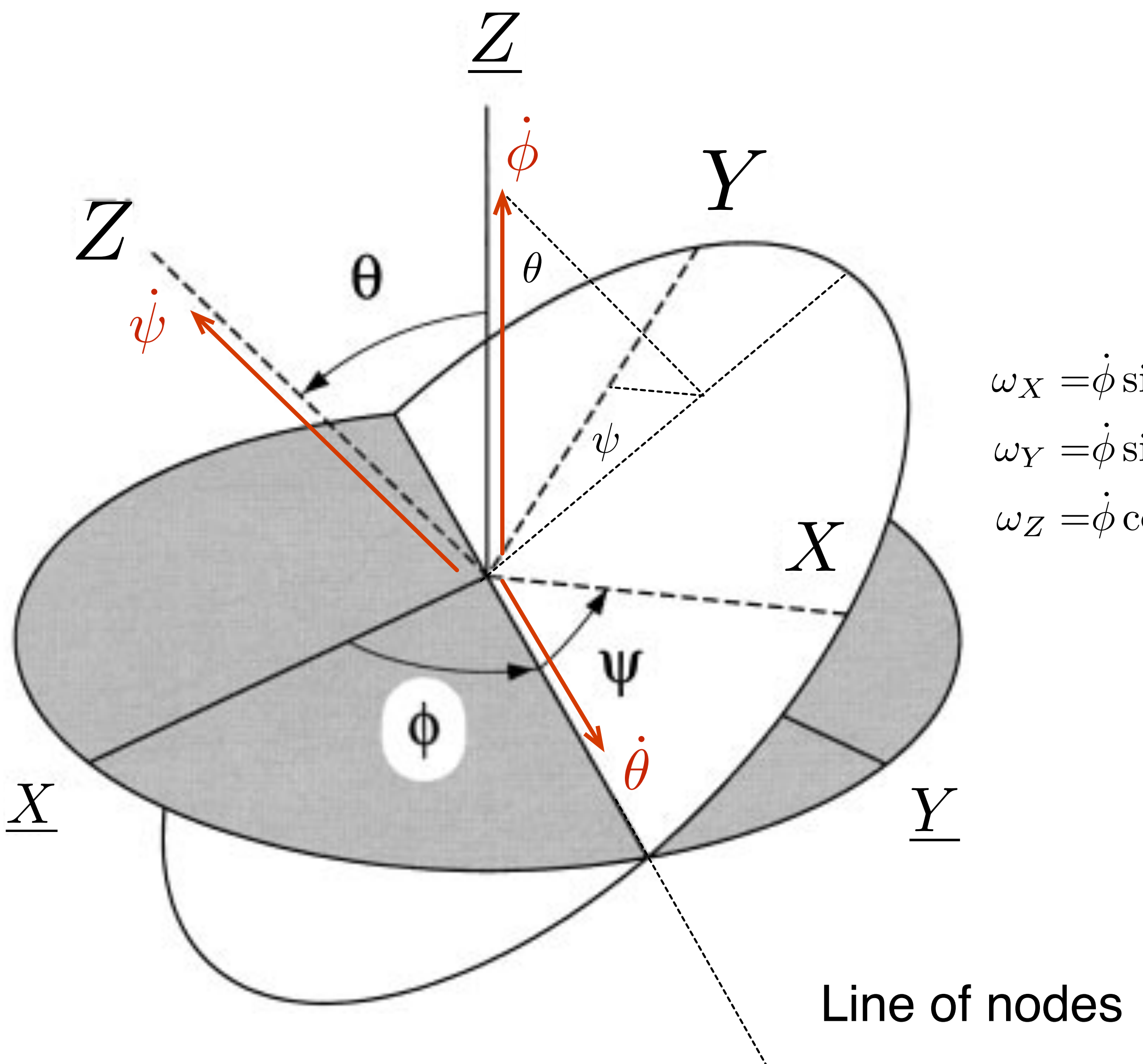
$$\omega_{z'} = \dot{\psi} + \dot{\phi}\cos\theta \qquad \text{(see picture) (below)}$$

The analogous formulas for $\omega_{x'}$ and $\omega_{y'}$ are given on the handout:

$$\boxed{\begin{cases} \omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_{z'} = \dot{\psi} + \dot{\phi}\cos\theta \end{cases}}$$

Angular velocity equations

These are a set of differential equations which can be integrated (numerically usually) to determine the orientation angles ψ, θ, ϕ



$$
\begin{aligned}
\omega_X &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\
\omega_Y &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\
\omega_Z &= \dot\phi \cos\theta \qquad\quad + \; \dot\psi
\end{aligned}
$$