Motion of a Heavy Symmetric Top

- We are considering a top which is spinning and symmetric (see picture on next page)

$$I_1 = I_2 \neq I_3$$

- We want to write down the Lagrangian of the system and compute the rate at which it precesses, i.e. compute $\dot{\phi}$

- We can either work with the CM as our origin, where the KE takes the form

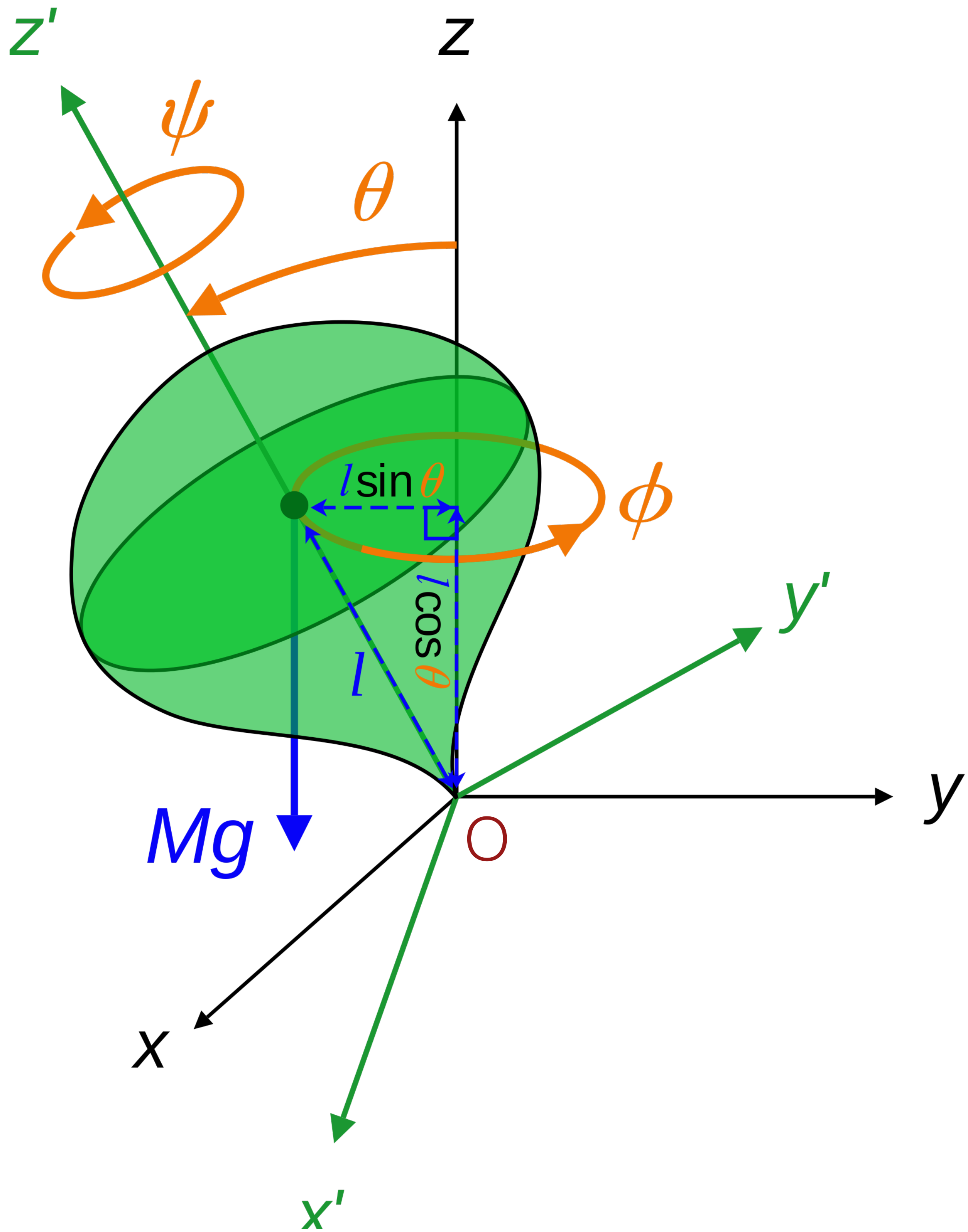
$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \omega_a I_{ab}^{cm} \omega_b$$

moving origin

- Or we can work with the fixed base, O, where the KE is described entirely by the rotational KE around this point

$$T = \frac{1}{2} \omega_a I_{ab}^{O} \omega_b$$

The moment of inertia needs to be calculated around this point. We will do this, and stop writing the I^O superscript, $I_{ab} \equiv I^O$, below.



z'
z
ψ
θ
l sin θ
ϕ
y'
l cos θ
l
y
Mg
O
x
x'

- The Lagrangian of the system is

$$L = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 - mgl\cos\theta$$

- Now we can express the Angular velocities in terms of the Euler Angles. Using the handout (see next page for picture)

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \frac{1}{2} I_3 (\dot{\psi} + \cos\theta\, \dot{\phi})^2$$

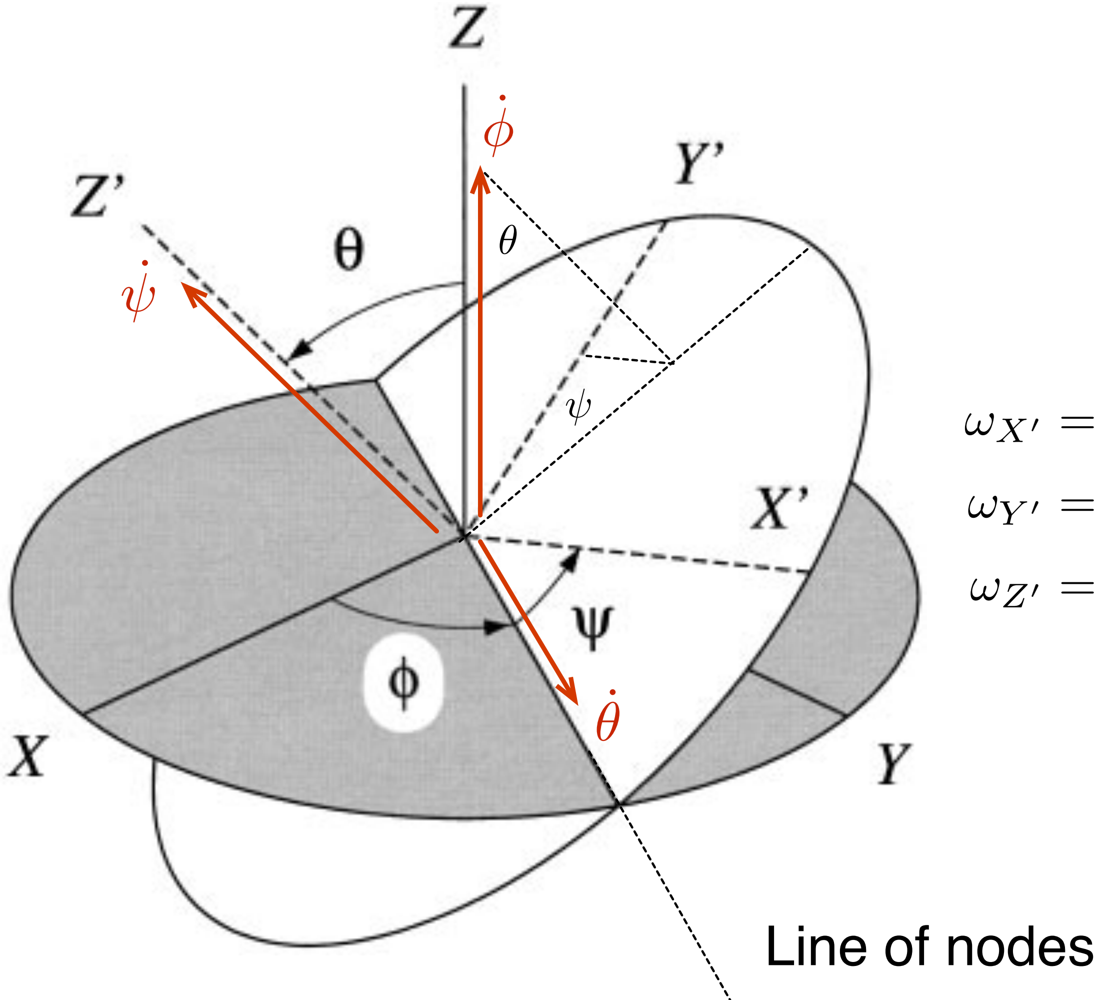
$$- mgl\cos\theta$$

- Now that we have this Lagrangian we can find the EOM. The external torque provided by gravity is perpendicular to the the $\dot{\psi}$ anxis and $\dot{\phi}$ axes and we therefore expect the angular momentum associated with these angles to be constant. Indeed they are cyclic coordinates

- $p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \cos\theta\, \dot{\phi})$ = constant in time since $\partial L / \partial \psi = 0$

- $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \sin^2\theta\, \dot{\phi} + I_3 (\dot{\psi} + \cos\theta\, \dot{\phi}) \cos\theta$ = const

$$= I_1 \sin^2\theta\, \dot{\phi} + I_3\, p_\psi \cos\theta$$



$$\omega_{X'} = \dot{\phi}\sin(\theta)\sin\psi + \dot{\theta}\cos(\psi)$$
$$\omega_{Y'} = \dot{\phi}\sin(\theta)\cos\psi - \dot{\theta}\sin(\psi)$$
$$\omega_{Z'} = \dot{\psi} + \dot{\phi}\cos(\theta)$$

- The two derivatives $\dot{\psi}$ and $\dot{\phi}$ can be expressed in terms of the constants p_ψ and p_ϕ

$$\boxed{\underset{\swarrow \text{precession rate}}{\dot{\phi}} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}} \quad , \quad \dot{\psi} = \frac{p_\psi}{I_3} - \dot{\phi}\cos\theta$$

- Note the various terms

- $$\frac{1}{2} I_1 \sin^2\theta\, \dot{\phi}^2 = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} \equiv \text{KE associated with } \dot{\phi}$$

And

- $$\frac{1}{2} I_3 (\dot{\psi} + \cos\theta\, \dot{\phi})^2 = \frac{p_\psi^2}{2I_3} \equiv \text{KE associated with spinning } \omega_{z'}$$

- Now we want to find an equation of motion for θ. Energy is constant

$$\boxed{\frac{1}{2} I_1 \dot{\theta}^2 + \overbrace{\left[mgl\cos\theta + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1} + \frac{p_\psi^2}{2I_3} \right]}^{\equiv\, U_{eff}(\theta)} = E}$$

This suggests that the equation of motion for θ is

$$I_1 \ddot{\theta} = -\frac{\partial U_{eff}}{\partial \theta} \qquad \text{verify yourself}$$

- This can formalized with the Routh procedure
The effective Lagrangian, $-R$, for the non-cyclic coordinate θ is

$$-R = L - p_\phi \dot{\phi} - p_\psi \dot{\psi}$$

Note that:

$$p_\phi \dot{\phi} + p_\psi \dot{\psi} = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3}$$

So

$$-R = \frac{1}{2} I \dot{\theta}^2 - U_{eff}(\theta)$$

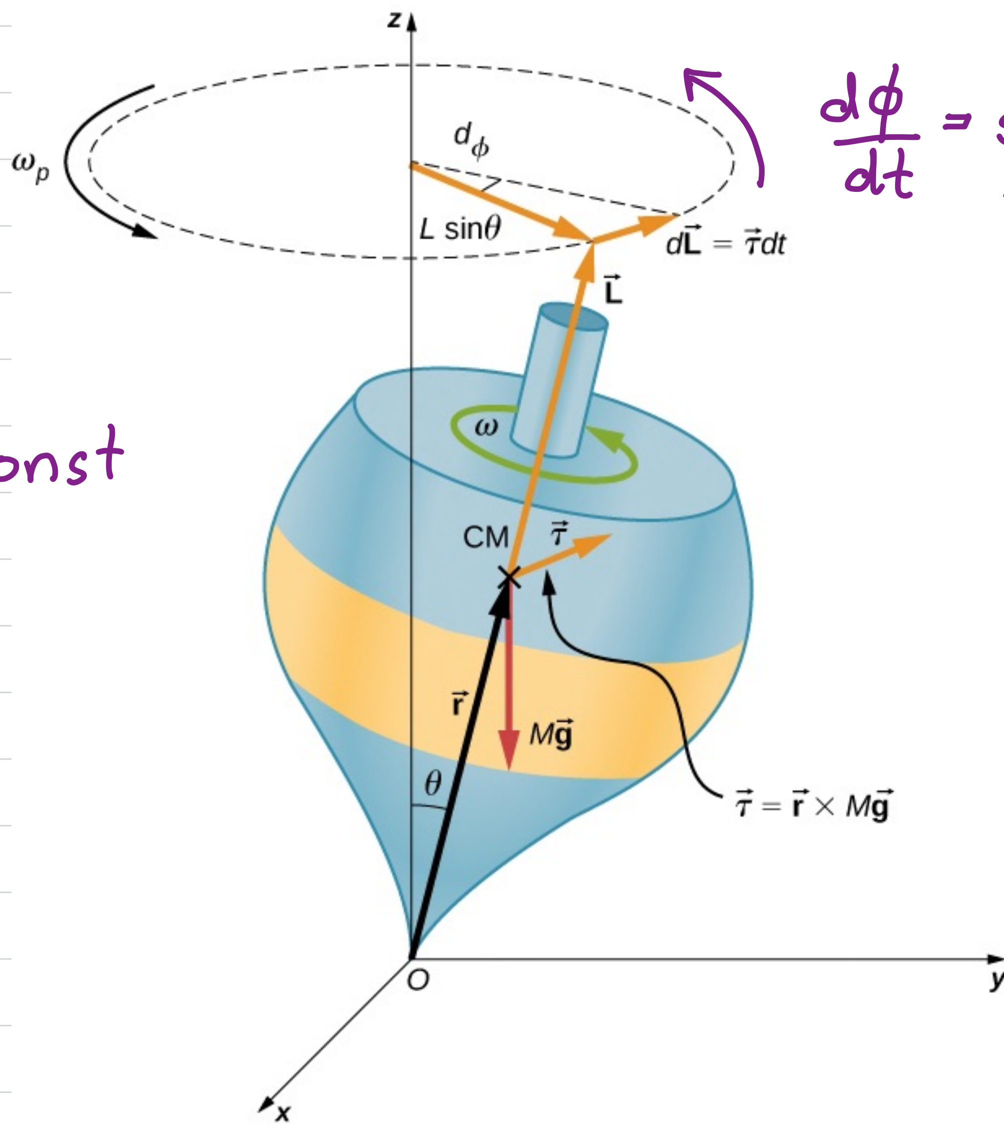
ignorable constant

Where $\boxed{U_{eff}(\theta) = mgl\cos\theta + \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta}} + \frac{p_\psi^2}{2I}$

Precessing Top Intro

- When the top is spinning quickly, there is a configuration where the angle θ is constant, and then the tip of the top precesses! Here we will analyze this configuration, with the Lagrangian setup.

- $\theta \approx$ const means we are at the minimum of $U_{eff}'(\theta)$, where $\ddot{\theta} \propto -\partial U_{eff}/\partial\theta = 0$!



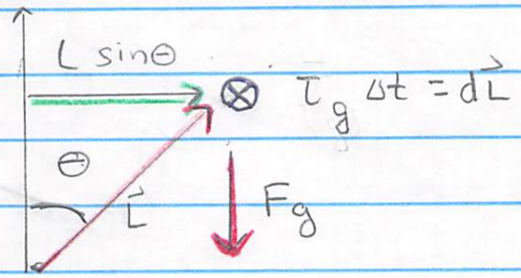
$\dfrac{d\phi}{dt}$ = slow precession rate

Θ = const

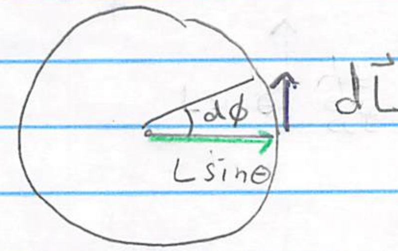
Now we will compute the precession rate i.e. $\dot{\phi}$. First the freshman physics way, and then using $U_{\text{eff}}(\theta)$

Analysis of precession rate of the top

- Side view (See figure) (next page)



Top view



The graviational torque is $\vec{\tau}_g = mgl\sin\theta$ (into page). So from geometry:

$$\frac{d\phi}{dt} = \frac{1}{L\sin\theta}\frac{dL}{dt} = \frac{mgl\,\cancel{\sin\theta}}{L\,\cancel{\sin\theta}}$$

$$\boxed{\frac{d\phi}{dt} \simeq \frac{mgl}{p_\psi}}$$

this assumes that θ doesn't change & $L \simeq p_\psi$

- This derivation is approximate assuming that the spinning KE along the body is large compared to the gravitational torque, saying $L \simeq p_\psi$.

- Now we derive $d\phi/dt$ from $U_{\text{eff}}(\theta)$, with the same approximations:

- $$U_{\text{eff}} = mgl\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta}$$

θ is constant at the min of $U_{\text{eff}}(\theta)$! Lets find θ_{min}

- $$\dot{\phi} = \frac{(p_\phi - p_\psi\cos\theta)}{I_1\sin^2\theta}$$ ← precession rate.

The rate at $\theta = \theta_{min}$ is our goal

- Lets switch to dimensionless variables:

$$\bar{g} \equiv \frac{mgl}{p_\psi^2/I} \equiv \text{gravitational torque relative to spinning KE}$$

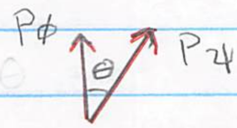
$$\beta \equiv P_\phi / P_\psi = \text{ratio of angular momenta}$$

So

$$\bar{U} \equiv \frac{U_{eff}}{p_\psi^2/I} = \bar{g}\cos\theta + \frac{(\beta-\cos\theta)^2}{2\sin^2\theta}, \qquad \dot\phi = \frac{P_\psi}{I} \frac{(\beta - \cos\theta)}{\sin^2\theta}$$

- For $\bar{g} \ll 1$ we can solve, for the minimum of U_{eff}. At zeroth order in gravity we neglect $\bar{g}$ and the potential is minimum (and θ is constant) at:

$$\cos\theta_{min} \simeq \beta$$



$$P_\phi \simeq P_\psi \cos\beta$$

- The precession rate $\dot\phi \propto (\beta - \cos\theta) = 0$. Of course! We need gravity for a torque. To first order

records

$$\cos\theta_{min} \equiv \beta + \delta$$ ↙ first order in $\bar{g}$, δ reords the shift in θ from gravity.

So $\sin^2\theta \simeq 1-\beta^2$ and U_{eff} is <u>approximately</u>:

$$U_{eff} = \bar{g}(\beta+\delta) + \frac{\delta^2}{2(1-\beta^2)}$$

which is minimized at $\delta = -\bar{g}\,(1-\beta^2)$

- The precession rate at Θ_{min} is:

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\Theta_{min}}{I_1 \sin^2\Theta_{min}} = \frac{p_\psi}{I_1} \frac{(\beta - \cos\Theta_m)}{\sin^2\Theta_m} \simeq \frac{p_\psi}{I_1} \frac{-\delta}{(1-\beta^2)}$$

$$\boxed{\dot{\phi} \simeq \frac{p_\psi}{I_1} \bar{g} \simeq \frac{p_\psi}{I_1}\left(\frac{mgl}{p_\psi^2/I_1}\right) = \frac{mgl}{p_\psi}.}$$

In agrement with the Freshman analysis, though now we see how to include corrections!

The "Sleeping" top

- The top is stable and spins in an upright position $\Theta = 0$. Lets analyze this case.

$$U_{eff}(\Theta) = mgl\cos\Theta + \frac{(p_\phi - p_\psi \cos\Theta)^2}{2I_1 \sin^2\Theta}$$

- Since $\Theta \simeq 0$ is stable it must be a min of $U_{eff}(\Theta)$

- $\Theta = 0$ can only be a stable point if $p_\phi \simeq p_\psi$. Then near $\Theta \simeq 0$, we expand:

$$\frac{(p_\phi - p_\psi\cos\Theta)^2}{2I_1\sin^2\Theta} = \frac{p_\psi^2}{2I_1}\frac{(1-\cos\Theta)^2}{\sin^2\Theta} \simeq \frac{p_\psi^2\,\theta^2}{8I_1}$$

- The potential near $\theta \simeq 0$ in this case is approximately, using $mgl\cos\theta \simeq mgl(1-\theta^2/2)$:

$$U_{eff} \simeq \text{const} + \left(\frac{p_\psi^2}{8I_1} - \frac{mgl}{2}\right)\theta^2$$

- So the potential is a minimum at $\theta = 0$ if

$$p_\psi > 2\sqrt{mgl\, I_1}$$

In terms of the spin rate $\omega_3 \simeq p_\psi / I_3$

$$\boxed{\omega_3 > 2\sqrt{\frac{mgl\, I_1}{I_3^2}}}$$

For ω_3 smaller than this, the top simply falls over. It is not spinning fast enough to stand up at $\theta = 0$.