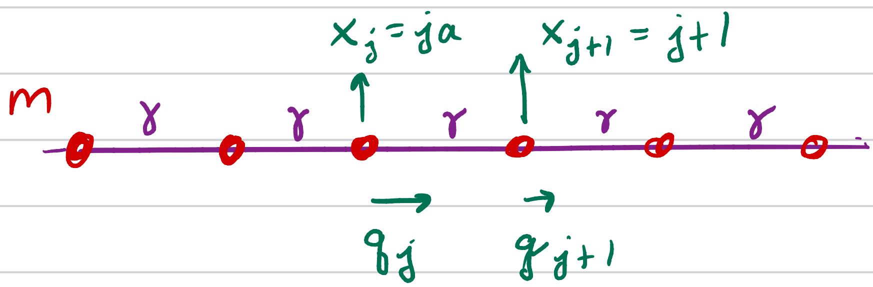
# Transition To Continuum

- Goldstein 13.1

- Consider an infinite chain of oscillators.
- The unperturbed positions are $x_j = ja$. The mass is $m$, the spring constant is $\gamma$ (since $k$ will be needed for wavenumber). The displacement of the $j$-th site is $q_j(t)$




The oscillators have a Lagrangian

$$L = \sum_j \frac{1}{2} m \dot{q}_j^2 - \tfrac{1}{2}\gamma (q_{j+1} - q_j)^2$$

with EOM

$$m\ddot{q}_\ell = \gamma(q_{\ell+1} - q_\ell) - \gamma(q_\ell - q_{\ell-1})$$

i.e

$$m\frac{d^2}{dt^2}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} -2\gamma & \gamma & & \\ \gamma & -2\gamma & \gamma & \\ & \gamma & -2\gamma & \gamma \\ & & \gamma & -2\gamma \end{pmatrix}$$

- As usual we look for eigenmodes

$$q_j = A\, E_j\, e^{-i\omega t}$$

eigenfrequency

- Substituting we have

$$\star \quad \underset{(1)}{-m\omega^2 E_j} \;\underset{(2)}{- \gamma(E_{j+1} - E_j)} + \underset{(3)}{\gamma(E_j - E_{j-1})} = 0$$

- Motivated by the wavelike Solution we take

$$\boxed{E_j = e^{ikx_j}\,, \quad \text{so} \quad q_j = A\, e^{ikx_j - i\omega t}}$$

eigenvector

- Now

$$(1) \quad q_{j+1} - q_j = A e^{-i\omega t}\left(e^{ikx_{j+1}} - e^{ikx_j}\right)$$

$$= A e^{-i\omega t + ikx_j}\left(e^{ika} - 1\right)$$

$$(2) \quad q_{j-1} - q_j = A e^{-i\omega t + ikx_j}\left(1 - e^{-ika}\right)$$

$$(3) \quad m\omega^2 q_j = A e^{-i\omega t + ik_j}$$

- So equation $\star$ yields

$$-m\omega^2 + \gamma\,(\underbrace{2 - e^{ika} - e^{-ika}}_{= 4\sin^2(ka/2)}) = 0$$

- Or

$$\omega_0 \equiv \sqrt{\frac{\gamma}{m}} \qquad \boxed{\omega^2(k) = 4\,\omega_0^2 \sin^2(ka/2)}$$

Debye Dispersion Curv

Summary:

① We have found a set of eigenvectors labelled by k. If only one eigenmode is excited

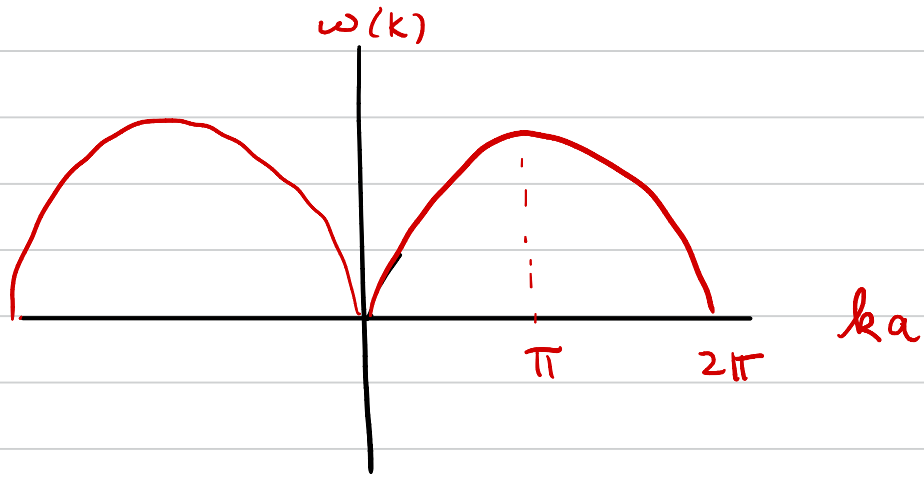
$$q_j(t) = \mathrm{Re}\left[A_k E_j^k e^{-i\omega t}\right]$$

$$= \mathrm{Re}\left[A_k e^{ikx_j - i\omega(k)t}\right]$$

eigenfrequency

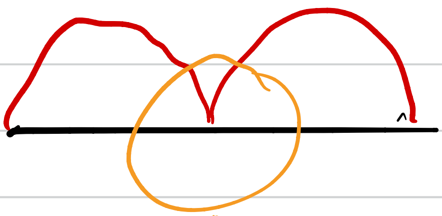
② The eigen-frequency for a given k is

$$\omega(k) = \pm 2\omega_0 \sin(ka/2)$$



# Transition to the Continuum

- In the low-$k$ limit we do not "see" the individual atoms. We should be able to reproduce our results with a continuum theory for $ka \ll 1$



region in detail

- Expanding $\sin(ka/2)$ for $ka \ll 1$ we have:

$$\omega = \pm 2\omega_0 \left( \frac{ka}{2} - \frac{1}{3!}\left(\frac{ka}{2}\right)^3 + \dots \right)$$

$$= \pm V_0 k \left( 1 - (ka)^2/24 + \dots \right)$$

where we have defined the velocity $\omega_0 a$

$$V_0 = \omega_0 a = a\sqrt{\frac{\gamma}{m}}$$

- The correction, $(ka)^2/24$, leads to dispersion and will change the velocity of the wave as $k$ becomes comparable to the spacing $a$, as we will see later

- First let's reproduce these results with continuum mechanics

# Transition to the continuum

- Returning to the lagrangian we introduce a **field** $q(t,x)$

$$q_j \longrightarrow q(t,x)$$ continuous variable: displacement at $x$ at $t$.

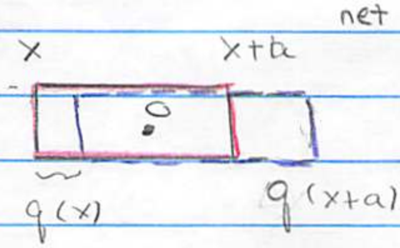
$$\sum \frac{m}{2} \dot{q}_j^2 \longrightarrow \int dx \, \mu \, (\partial_t q)^2$$

mass per length $= m/a$

- The potential term is

Young modulus $\equiv Ka$

$$\sum_j a (Ka) \left( \frac{q_{j+1} - q_j}{a} \right)^2 \longrightarrow \int dx \, Y \left( \frac{\partial q}{\partial x} \right)^2$$




net force on O:

$$F_O = Y \, \xi$$

elongation per length

$$= Y \left( \frac{q(x+a) - q(x)}{a} \right)$$

So the Lagrangian and action:

Lagrangian and action of Rod:

$$L = \int dx \; \frac{1}{2} \mu (\partial_t q)^2 - \frac{1}{2} Y (\partial_x q)^2$$

Lagrange density

$$S = \int dt \, dx \left[ \frac{1}{2} \mu \, (\partial_t q)^2 - \frac{1}{2} Y (\partial_x q)^2 \right]$$

# The Lagrange Equations

- Now lets determine the equation of motion

$$S[q] = \int dt\, dx\, \mathcal{L}(\partial_t q, \partial_x q, q)$$

lagrange density

- Lets get a notation:

$$\left.\begin{aligned} y^a &= (y^1, y^2) = (t, x) \\ \partial_a q &= (\partial_t q, \partial_x q) \\ dt\,dx &= d^2 y \end{aligned}\right\}$$

So

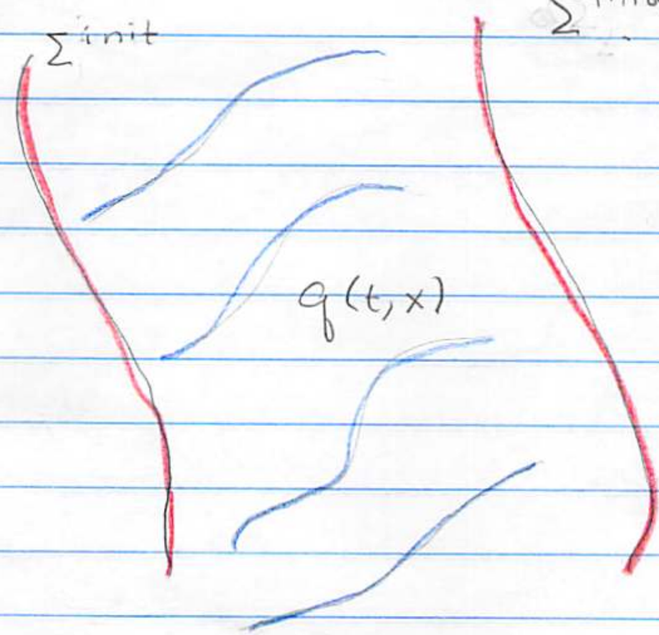
$$\mathcal{L}(\partial_a q, q)$$

$$P^a = \frac{\partial \mathcal{L}}{\partial(\partial_a q)} = \text{canonical momentum}$$

$$\equiv (P^0, P^1)$$

- The variational problem takes place between two space time surfaces

$\Sigma_1 = \Sigma^{init}$ $\qquad$ $\Sigma^{final} = \Sigma_2$




$\Sigma_1, \Sigma_2$ are analogous to the endpoints $t_1, t_2$

Varying the action $S[q+\delta q] - S[q] \equiv \delta S$

$$\delta S = \int d^2y \; \underline{\underline{\frac{\partial \mathcal{L}}{\partial(\partial_a q)} \partial_a(\delta q)}} + \frac{\partial \mathcal{L}}{\partial q} \delta q$$

So we integrate by parts:

$$= \partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a q)} \delta q \right) - \partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a q)} \right) \delta q$$

divergence of a vector: $\partial_a P^a = \vec{\nabla} \cdot \vec{V}$

So using the divegence theorem:

$$\delta S = \underbrace{\int_{\Sigma_2} d\Sigma_{2a} P^a \delta q - \int_{\Sigma_1} d\Sigma_{1a} P_1^a \, \delta q(1)}_{\text{bndry terms}}$$

$$+ \int d^2y \left( -\partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a q)} \right) + \frac{\partial \mathcal{L}}{\partial q} \right) \delta q$$

So the continuum Lagrange eom read

$$\boxed{-\partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a q)} \right) + \frac{\partial \mathcal{L}}{\partial q} = 0}$$

← Euler Lagrange equations for a field

Compare

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0$$

- For the current case:

$$\mathcal{L} = \frac{1}{2}\mu(\partial_t q)^2 - \frac{1}{2}Y(\partial_x q)^2 \quad \leftarrow \text{Lagrange density}$$

- The EOM read:

$$-\partial_t\left(\frac{\partial \mathcal{L}}{\partial(\partial_t q)}\right) - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x q)} = 0$$

$$-\partial_t(\mu\,\partial_t q) + \partial_x(Y\partial_x q) = 0$$

- Leading to a wave equation for $q$

$$\boxed{\mu\,\partial_t^2 q - Y\,\partial_x^2 q = 0}$$

If we substitute $\boxed{q = a\,e^{ikx}\,e^{-i\omega(k)t}}$ we find

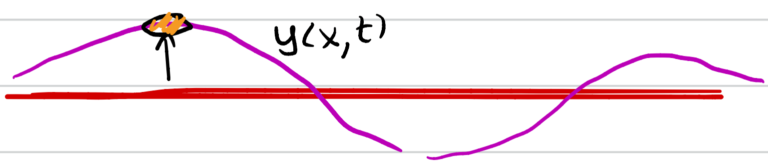
$$-\mu\,\omega^2 + Y k^2 = 0$$

$$\omega(k) = \pm\sqrt{\frac{Y}{\mu}}\,k$$

$$\boxed{\omega(k) = \pm v_0 k}$$

## Another Example : The string

- Consider the string with tension $T$




- The Kinetic Energy is $\mu \equiv$ mass / Length

$$KE = \int dx \; \frac{1}{2} \mu (\partial_t y)^2$$

- The potential energy arises due to the stretching. We are increasing the length from $\ell_1 = \Delta x$ to $\ell_2 = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. This takes work which is

$$\Delta W = \int_{\ell_1}^{\ell_2} T \, d\ell = T \left[ \sqrt{\Delta x^2 + (\Delta y)^2} - \Delta x \right]$$

$$\simeq \frac{T}{2} \left( \frac{\partial y}{\partial x} \right)^2 \Delta x$$

expand for small $\Delta y / \Delta x \ll 1$

- So

$$S = \int dt\, dx \left[ \frac{1}{2} \mu (\partial_t y)^2 - \frac{T}{2} (\partial_x y)^2 \right]$$

- Thus we can find the EOM by varing the action, e.g.

$$(\partial_t y)^2 \longrightarrow (\partial_t y + \partial_t \delta y)^2 \simeq (\partial_t y)^2 + 2\,\partial_t y\, \partial_t \delta y$$

So

$$\delta S = \int dt\, dx \left[ \mu\, \partial_t y\, \partial_t \delta y - T\, \partial_x y\, \partial_x \delta y \right]$$

integrate by parts

And

$$\delta S = \int dt\, dx\, \underline{\underline{\left[ -\partial_t(\mu\, \partial_t y) + \partial_x (T\, \partial_x y) \right]}}\, \delta y = 0$$

Setting $\underline{\underline{\quad\quad}}$ to zero gives the EOM.