# Waves on the string

- Take the wave equation

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

- Then we find specific solution $y(t,x) = \mathrm{Re}\left[A e^{ikx - \omega(k)t}\right]$

with

$$\omega(k) = \pm vk \quad \text{from} \quad -\frac{\omega^2(k)}{v^2} + k^2 = 0$$

- Then the general solution is a sum of these

$$y(t,x) = \mathrm{Re}\left[\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\tilde{A}(k)\,e^{ik(x-vt)} + \tilde{B}(k)\,e^{-ik(x+vt)}\right]$$

$$= A(x-vt) + B(x+vt)$$

right mover　　left mover

- Lets evaluate the energy in the string:

$$\bar{\mathcal{E}} = \overline{\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2}$$

bar indicates a time average

For a plane wave

# Averaging over Time

- Then for harmonic quantites

$$A(t) = \mathrm{Re}\left[A_\omega e^{-i\omega t}\right] \qquad B(t) = \mathrm{Re}\left[B_\omega e^{-i\omega t}\right]$$

- Then

$$\overline{A(t)B(t)} = \frac{1}{2}\left(A_\omega e^{-i\omega t} + A_\omega^* e^{i\omega t}\right) \times$$

$$\frac{1}{2}\left(B_\omega e^{-i\omega t} + B_\omega^* e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(A_\omega B_\omega^* + A_\omega^* B_\omega\right) + \underbrace{\sim e^{-2i\omega t} \text{ and } e^{+i2\omega t}}_{\text{these are oscillating}}$$

- Thus

$$\boxed{\overline{A(t)B(t)} = \frac{1}{2}\mathrm{Re}\left(A_\omega B_\omega^*\right)}$$

and so

$$\overline{A^2(t)} = \frac{|A_\omega|^2}{2}$$

So for $y = Ae^{ikx - i\omega t}$ $\quad \partial y/\partial x = ikAe^{ikx - i\omega t}$

$$\overline{\left(\frac{\partial y}{\partial x}\right)^2} = \frac{|ikA|^2}{2} = \frac{k^2|A|^2}{2}$$

- So similarly

$$\overline{\mathcal{E}} = \overline{\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2}$$

$$= \frac{1}{2}\mu w^2 A^2 + T k^2 A^2 \qquad T k^2 = \mu w^2$$

Since $V^2 = \frac{T}{\mu}$

$$\boxed{\overline{\mathcal{E}} = \frac{1}{2}\mu w^2 A^2}$$

average energy / length in a plane wave

- And then notice the force,

$F = T\frac{\partial y}{\partial x}$, is in phase with the velocity $\frac{\partial y}{\partial t}$. So:

definition of impedance (and why we care)

$$\boxed{\frac{F}{\partial y/\partial t} = \frac{T\,ik}{-iw} = \pm\sqrt{T\mu}}$$

$\sqrt{T\mu} \equiv$ impedace $Z$

= ratio of force term to velocity term.

- The power is

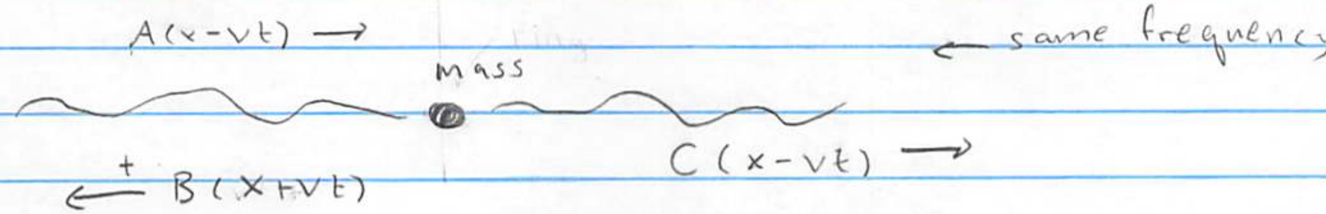
$$\overline{S^x} = \overline{-T\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}} = \frac{Tkw}{2}|A|^2 = \boxed{\frac{Z}{2}w^2|A|^2 = \overline{S^x}}$$

$$T\frac{\partial y}{\partial x} = Z\frac{\partial y}{\partial t}$$

average power transmitted

# Transmission and Reflection

- Wave $\mu_1, k_1$ Wave $\mu_2$ $k_2$




- A wave $A(x-vt)$ comes in and gets reflected by a mass of mass $m$. Determine the reflected and transmitted wave forms given $A(x-vt)$

- Lets work in fourier space:

- The solution to the left reads

$$y(t,x) = \tilde{A}_k e^{ik_1x - i\omega t} + \tilde{B}_k e^{-ik_1x - i\omega t}$$

While to the right

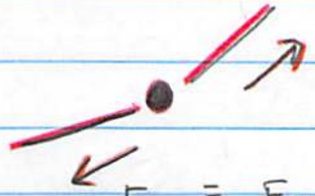
$$y(t,x) = \tilde{C}_k e^{ik_2x - i\omega t}$$

- We need to determine $\tilde{C}_k$ and $\tilde{B}_k$ given $\tilde{A}_k$

- Demanding continuity

☆ $\tilde{A} + \tilde{B} = \tilde{C}$

A free body diagram of the mass is



$F_R^y \equiv$ Force by right on ring $T\frac{\partial y}{\partial x}$

$F_L \equiv$ Force by left on ring $-T\frac{\partial y}{\partial x}$

So we have:

$$F_R^y + F_L^y = m\ddot{y}$$

$$T\frac{\partial y_R}{\partial x} - T\frac{\partial y_L}{\partial x} = m\ddot{y}$$

$T\,\partial y/\partial x = Z\,\partial y/\partial t$

note; $Tk = Z\omega$

So in fourier space

$$T i k_2 C - T i k_1 (A - B) = -m\omega^2 C$$

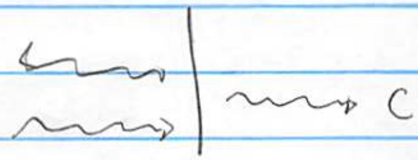
★★ $$i\omega Z_2 C - i\omega Z_1 (A - B) = -m\omega^2 C$$

Solving Eq ★★ and Eq ★ for C/A and B/A:

$$r(k) \equiv \frac{B}{A} = -\frac{m\omega + i(Z_1 - Z_2)}{m\omega - i(Z_1 + Z_2)}$$

$$t(k) \equiv \frac{C}{A} = -\frac{2iZ_1}{m\omega - i(Z_1 + Z_2)}$$

- First lets check that the power is conserved




- The incoming energy per time is

$$P_{in} = S^x_A = Z_1 \frac{\omega^2}{2} |A|^2$$

- While the outgoing powers are

$$S^x_B = Z_1 \frac{\omega^2}{2} |B|^2$$

$$S^x_C = Z_2 \frac{\omega^2}{2} |C|^2$$

note:

- So need to have

$$\boxed{S^x_A = S^x_B + S^x_C} \Rightarrow \boxed{1 = |r|^2 + |t|^2 \frac{Z_2}{Z_1}}$$

divide by $S^x_A$

You can check that this is the case given the $r(\omega)$ and $t(\omega)$ coefficients on the previous page

- Now lets explore the solution <u>in space.</u> For this part set $\mu_1 = \mu_2$

$$\left.\begin{aligned} \tilde{B}(k) &= \tilde{A}(k)\, r(k) \\ \tilde{C}(k) &= \tilde{A}(k)\, t(k) \end{aligned}\right\}$$

so $z_1 = z_2$ from now on.

- I will also return to $k$, $\omega = vk$, define

$$k_0 = \frac{z_1 + z_2}{mv} = \frac{2\mu}{m}$$

notice the limit $m \to \infty$ means $k_0 \to 0$

- So

$$r(k) = -\frac{k}{k - ik_0} \xrightarrow[m\to\infty]{} -1$$

$$t(k) = \frac{-ik_0}{k - ik_0} \xrightarrow[m\to\infty]{} 0$$

- And thus, for example, the transmitted wave is

$$\boxed{C(x - vt) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, A(k)\, t(k)\, e^{ik(x - vt)}}$$

This is a product in Fourier space, which in coordinate space becomes a convolution

- So defining $\xi \equiv x - vt$ (comoving coordinate

$$C(x-vt) = \int_{-\infty}^{\infty} dx' \, A(x') \, t(\xi - x')$$

Where the transmission kernel is $t(k) = -k_0/(k - ik_0)$

$$\tau(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, t(k) \, e^{ikx} = k_0 \, e^{-k_0 x'} \, \Theta(-x')$$

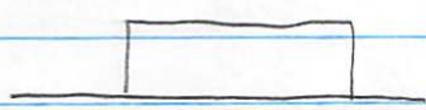
do integral

- Thus the outgoing wave takes the waveform which comes in and linearly smears with a transformation kernel to produce the outgoing wave

<u>For example;</u>

- Take a square pulse of width $a$

$$A(x-vt) = \Theta(a - |x-vt|)$$



incoming

- The transmission kernel will smear over a length $k_0$ leading to distorted shape

$\longrightarrow C(x-vt)$