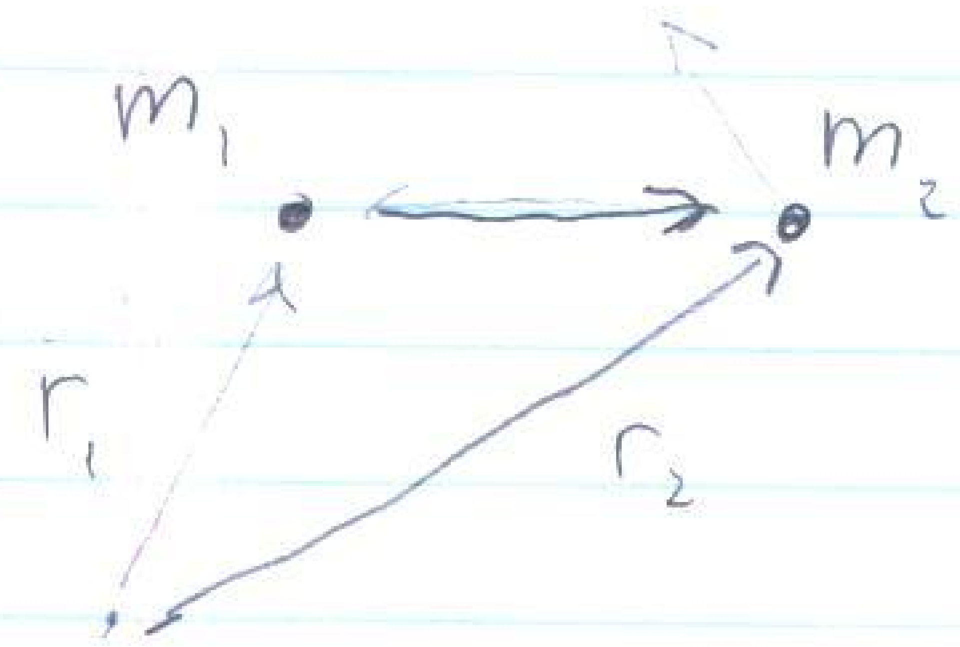
Motion in a spherical potential

$V(r)$

Classical Motion, Relative and CM coordinates



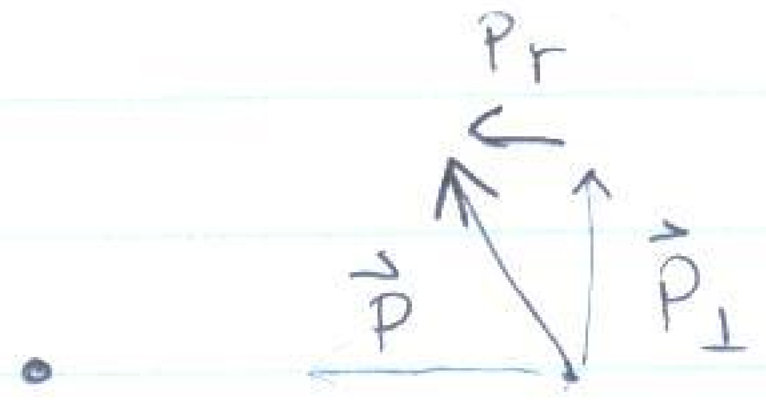
$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \vec{P}_{cm} = m_1 \vec{V}_1 + m_2 \vec{V}_2$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{P}_{rel} = m_{red} (\vec{V}_2 - \vec{V}_1)$$

$$m_{red} = \frac{m_1 m_2}{m_1 + m_2}$$

$$KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$$

$$KE = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} = \frac{P_{cm}^2}{2M_{Tot}} + \frac{P_{rel}^2}{2m_{red}}$$



$$KE = \frac{p_r^2}{2m} + \frac{p_\perp^2}{2m} = \frac{p_r^2}{2m} + \frac{(r p_\perp)^2}{2mr^2}$$

$$KE = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} \qquad |\vec{r} \times \vec{p}| = r p_\perp = |\vec{L}|$$

Then Look at quantum mechanics

$$KE = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2$$

Look for: Eigen Funcs

$$\left[-\frac{\hbar^2}{2m} \nabla_r^2 + V(r) \right] \phi_E = E \phi_E$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2}$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \begin{vmatrix} \hat{i} & x & p_x \\ \hat{j} & y & p_y \\ \hat{k} & z & p_z \end{vmatrix}$$

$$L_x = y p_z - z p_y$$

$$L_y = -x p_y + y p_x$$

$$L_z = x p_y - y p_x = x \, -i\hbar \frac{\partial}{\partial y} - y \, -i\hbar \frac{\partial}{\partial x}$$

$$\xrightarrow[\text{polar cords}]{} -i\hbar \frac{\partial}{\partial \phi}$$

$$L^2 \xrightarrow[\text{polar cords}]{} -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial\phi^2} \right]$$

Then the Schrödinger Equation

$$\left[\frac{-\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2mr^2} + V(r) \right] \phi_E(r,\theta,\phi)$$

$$= E\,\phi_E(r\theta,\phi$$

Suppose we can find eigenstates of L^2

$$L^2\, Y(\theta,\phi) = C\; Y(\theta,\phi)$$

$$L^2$$

This is a second order differential equation
can go find the solution

- Always can find a solution
- Only for particular values of C is that solution bounded

$$L^2\, Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi) \qquad l = 0,1,2,3\ldots$$

$$L_z\, Y_{lm}(\theta,\phi) = m\hbar\, Y_{lm}(\theta,\phi) \qquad m \le l$$

Then we return to the Schrödinger equation

$$(1.1)\quad \left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2mr^2} + V(r)\right]\phi_E(r,\theta,\phi) = E\,\phi_E(r,\theta,$$

Write

$$\boxed{\phi_E = R_{lm}(r)\, Y_{lm}(\theta,\phi)} \qquad \int_0^\infty dr\, r^2 |R_{lm}(r)|^2 = 1$$

$$L^2\, r\,u(r)\, Y_{lm} = l(l+1)\hbar^2\, r\,u(r)\, Y_{lm}$$

Then Eq (1.1) becomes

$$(1.2)\quad \left[\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + V(r)\right] R_{lm}(r) = E\, R_{lm}(r$$

Make substitution

$$R_{lm} = \frac{u_{lm}}{r}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{u}{r}\right) = \frac{1}{r}\frac{\partial^2 u}{\partial r^2}$$

Then Eq (1.2) becomes

$$\left[\underbrace{\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial r^2}} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r)\right] u_l(r) = E\, u_l(r)$$

(1.3) $$\left[\frac{P_r^2}{2m} + \frac{l(l+1)\hbar^2}{2mr^2} + V(r)\right] u_l(r) = E\, u_l(r)$$

The norm condition

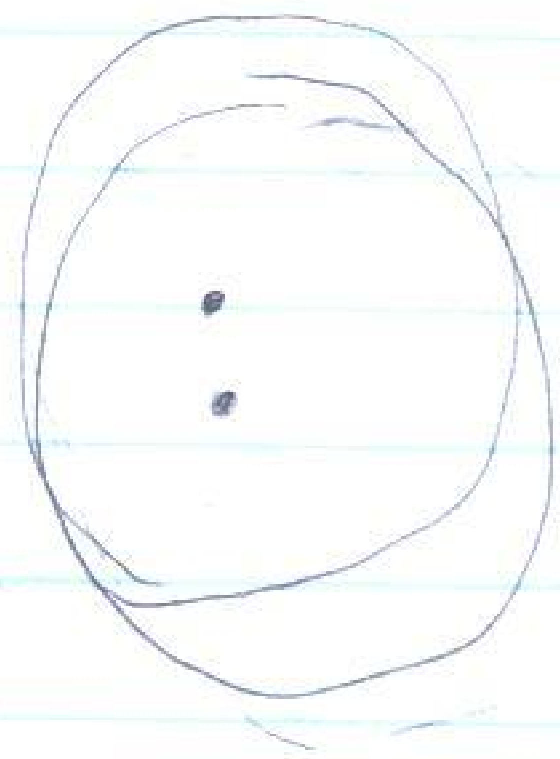
$\int dr\, r^2 |R_{lm}(r)|^2 = 1$ becomes $\int_0^\infty dr\, (u_l(r))^2 = 1$

Equation (1.3) is like a radial Schrödinger equation for $u(r)$.

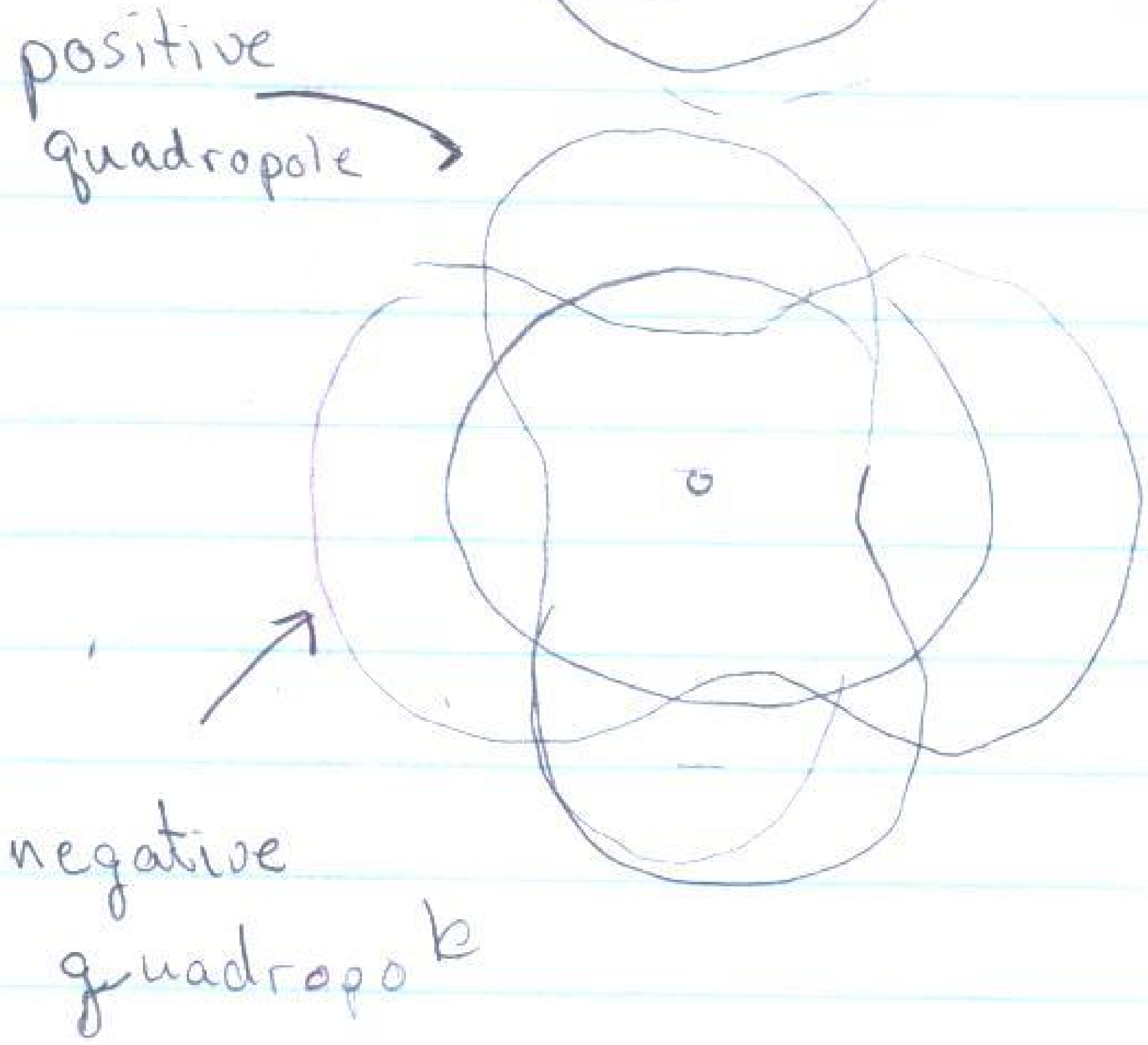
① $R_{lm}(r) = \frac{u_{lm}(r)}{r}$ need $u_{lm}(r) \xrightarrow[r\to 0]{} 0$ linearly

② The potential is $V(r) + \overbrace{\frac{l(l+1)\hbar^2}{2mr^2}}^{\text{centrifugal barrier}}$

$l = 0$



$l = 1$ dipole



$l = 2$ quadropole

http://www.bpreid.com/applets/poasDemo.html

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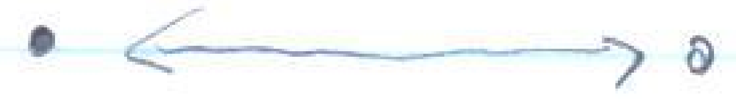

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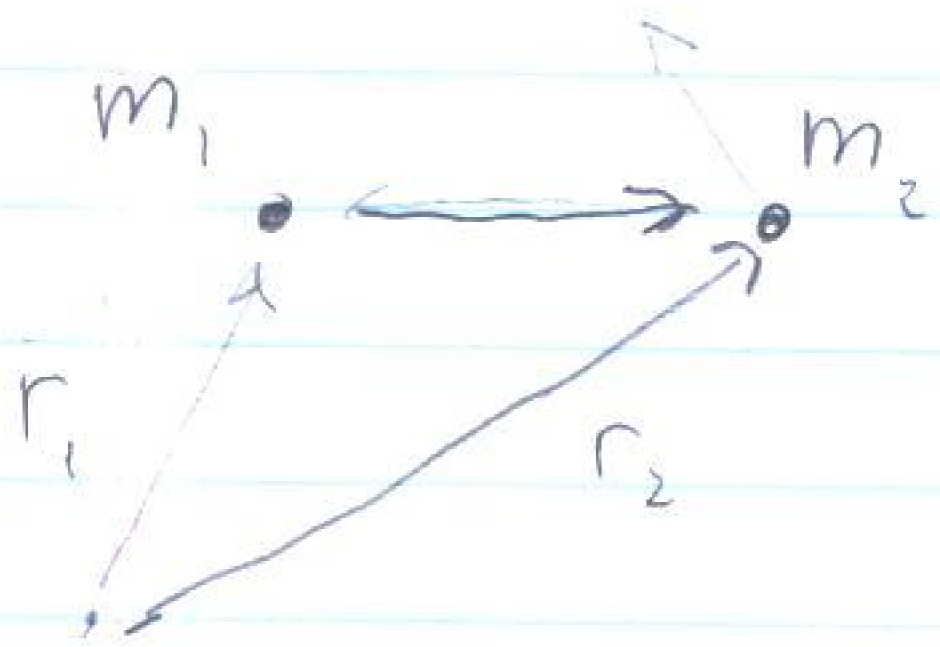

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