Quantum Mechanics

- Particles are waves $\psi(x,t)$

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- Position and momentum are operators which act on waves

$$\hat{P}\,\psi(x,t) = -i\hbar \frac{\partial}{\partial x}\psi(x,t)$$

$$\hat{X}\,\psi(x,t) = x\,\psi(x,t)$$

  - operators take waves and spit out transformed waves

- The wave function changes in time, physics predicts this change

$$\underbrace{\left(\frac{\hat{P}^2}{2m} + \hat{V}(x)\right)}_{\text{The Hamiltoniam} = KE + PE}\psi(x,t) = i\hbar\frac{\partial \psi(x,t)}{\partial t}$$

The schrödenger equation:

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How to Find the average of an operator:

$\langle \hat{x} \rangle$ = average position of particle

$$= \int dx\ \psi^*(x,t)\ \hat{X}\ \psi(x,t)$$

$$\langle \hat{x} \rangle = \int dx\ x\ |\psi|^2$$

$\langle \hat{P} \rangle$ = Averag momentum of a particle

$$\langle \hat{p} \rangle = \int dx\ \psi^*(x,t)\ \hat{P}\ \psi(x,t)$$

$$= \int dx\ \psi^*(x,t)\ -i\hbar \frac{\partial}{\partial x} \psi(x,t)$$

"Eigenstates" of Operators:

$$\hat{\Omega}\, \phi_\omega(x) \propto \phi_\omega(x) \qquad \text{or} \qquad \hat{\Omega}\, \phi_\omega(x) = \omega\, \phi_\omega(x)$$

- certain functions that when you act with an operator you get back the same function times a number

- Example #1, Eigenstates of the Hamiltonian

$$\hat{H}\, \phi_E(x) = E\, \phi_E(x)$$

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \phi_E(x) = E\, \phi_E(x)$$

Also known as the Schrödinger equat

- Example #2, Eigenstates of momentum

$$\hat{P}\, \phi_p(x) = p\, \phi_p(x)$$

$$-i\hbar \frac{\partial}{\partial x} \phi_p(x) = p\, \phi_p(x)$$

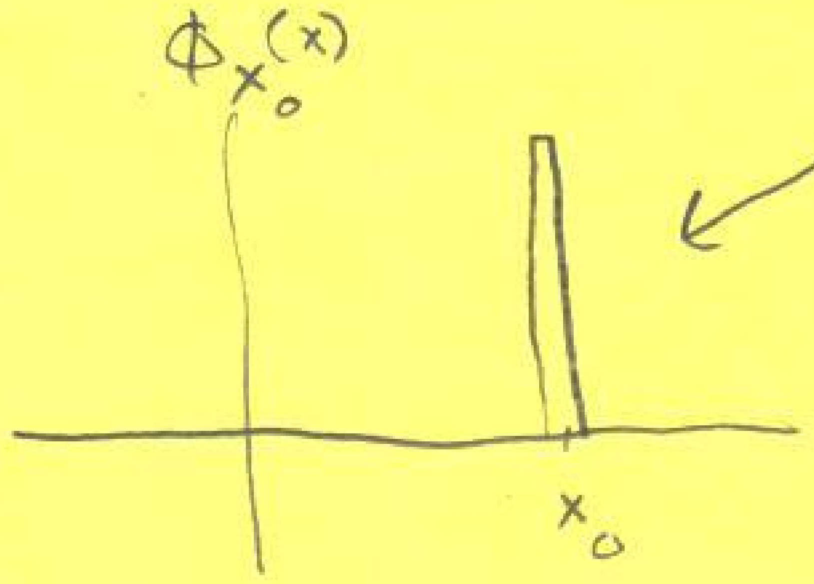
Differential equation for $\phi_p(x)$

So $\phi_p(x) = C e^{-ip\cdot x/\hbar}$

we will come back to the normalization in a bit

Example 3

- Eigenstates of $\hat{X}$



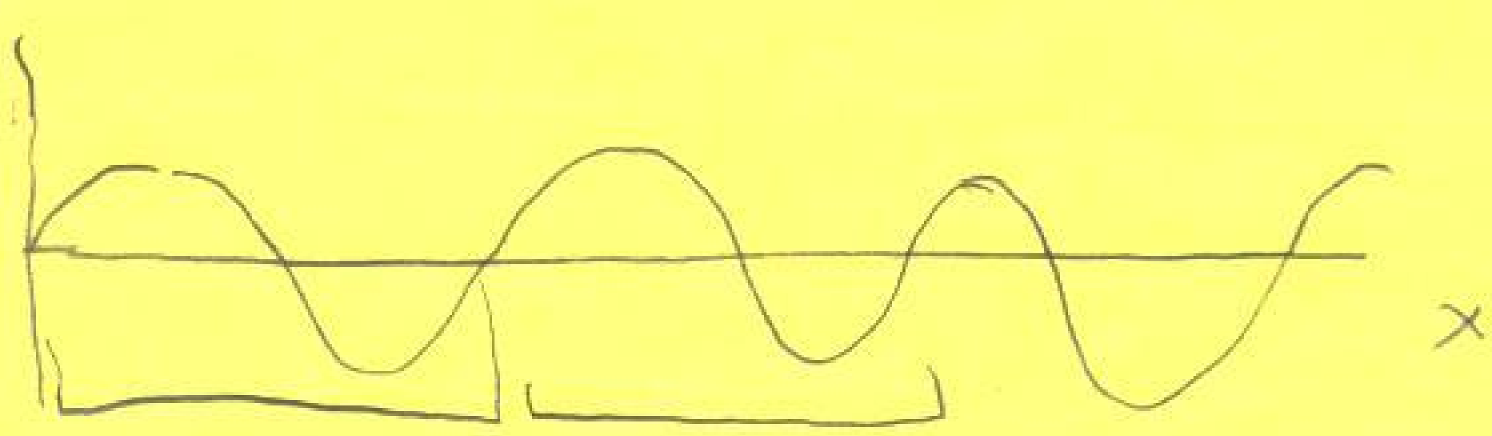
Very sharply peaked function at x_0

$$\hat{X}\,\phi_{x_0}(x) = x\,\phi_{x_0}(x) \approx x_0\,\phi_{x_0}(x)$$

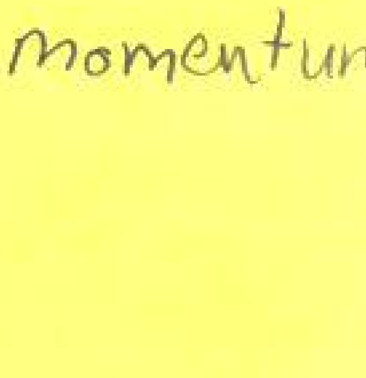
<u>Then</u>

- Take position $\hat{X}$, functions which are eigenstates $\phi_{x_0}(x)$ have definite position, x_0.

- Take momentum $\hat{P}$, $\phi_p(x) = C e^{i p_0 \frac{x}{\hbar}}$



(most functi
don't)



$\phi_p(x)$ has definite wave length λ

$\lambda = \dfrac{2\pi\hbar}{p}$; definite wavelength = definite momentum

de Broglie

$\lambda = \dfrac{h}{p}$

Why do we care about Energy Eigenstates?

Suppose that at time $t=0$ we are in a energy eigenstate $\phi_E(x)$

$$\psi(x,t_0) = \phi_E(x)$$

Then at later times we try to solve the Schrödinger Equation

$$\hat{H}\,\psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t}$$

Ansatz:

$$\psi(x,t) = \phi_E(x)\,C(t) \qquad C(t_0) = 1$$

$$\hat{H}\,\psi(x,t) = i\hbar \frac{\partial \psi}{\partial t}$$

$$\hat{H}\,\phi_E\,C(t) = i\hbar\,\phi_E\,\partial_t C$$

$$E\phi_E\,C(t) = i\hbar\,\phi_E\,\partial_t C(t)$$

$$E\,C(t) = i\hbar\,\partial_t C(t)$$

Solution $\quad C(t) = e^{-\frac{iE}{\hbar}(t-t_0)}$

$$\psi(x,t) = \phi_E(x)\, e^{-iE\Delta t/\hbar}$$

$$|\psi^*(x,t)|^2 = \underbrace{\phi_E^*\, e^{+iE\Delta t/\hbar}}_{\psi^*(x,t)} \; \underbrace{\phi_E(x)\, e^{-iE\Delta t/\hbar}}_{\psi(x,t)}$$

$$= \phi_E^*(x)\, \phi_E(x)$$

$$|\psi(x,t)|^2 = |\phi_E(x)|^2$$

independent of time

Energy eigenstates are stable don't change in time

Fourier Series:

Consider a Periodic Function

Any periodic function can be expanded in sin's and cos's

Completeness

$$f(x) = C_0 + B_1 \sin \frac{2\pi}{L} x + B_2 \sin \frac{2\pi \cdot 2x}{L} + B_3 \sin \frac{2\pi}{L} 3x + \ldots$$

$$+ A_1 \cos \frac{2\pi}{L} x + A_2 \cos \frac{2\pi}{L} \cdot 2x + A_3 \cos \frac{2\pi}{L} 3x \ldots$$

$$\boxed{f(x) = C_0 + \sum_{k>0} A_k \cos kx + B_k \sin kx \qquad k = \frac{2\pi}{L} n \quad n = 1 \ldots, \infty}$$

$$\cos kx = \frac{e^{+ik\cdot x} + e^{-ikx}}{2} \qquad \sin kx = \frac{e^{+ikx} - e^{-ikx}}{2i}$$

$$= C_0 + \sum_{k>0} \left(\frac{A_k - iB_k}{2}\right) e^{+ik\cdot x} + \left(\frac{A_k + iB_k}{2}\right) e^{-ik\cdot x}$$

$$C_k = \frac{A_k - iB_k}{2} \quad k>0 \qquad C_{-k} = \frac{A_{|k|} + iB_{|k|}}{2}$$

Then we have

$$f(x) = C_0 + \sum_{k>0} C_k e^{ik \cdot x} + \sum_{k>0} C_{-k} e^{-ikx}$$

$$\boxed{f(x) = \sum_k C_k e^{ikx} \qquad k = \frac{2\pi}{L} n \quad n = -\infty \ldots \infty}$$

How to find the coefficients:

- Orthogonality:

$$\int_{L/2}^{L/2} e^{-i\frac{2\pi}{L}n'} e^{+i\frac{2\pi}{L}n} = 0 \quad \text{for } n \neq n'$$

because it oscillates

$$= L \quad \text{for } n = n'$$

So

$$\int_{-L/2}^{L/2} e^{-ik \cdot x} f(x) = \int_{-L/2}^{L/2} \left(\sum_{k'} C_{k'} e^{+ik'x}\right) e^{-ikx}$$

$$= \sum_{k'} \delta_{kk'} L C_k = C_k L$$

$$\boxed{C_k = \frac{1}{L} \int_{-L/2}^{L/2} e^{-ikx} f(x)}$$

We will rewrite ~~this~~ this:

- $$\Phi_k = \frac{e^{ik\cdot x}}{\sqrt{L}} \qquad \int_{-L/2}^{L/2} \phi^*_{k'} \phi_k = \delta_{kk'}$$

- $$f(x) = \sum_k f(k)\, \Phi_k$$

- $$\int_{-L/2}^{L/2} dx\, \Phi^*_k\, f(x) = f(k) \qquad f(k) = \sqrt{L}\, C_k$$

Can also do it in terms of sin's and cos's:

$$C_k = \frac{1}{L} \int_{-L/2}^{L/2} e^{-ikx} f(x)$$

$$C_{-k} = \frac{1}{L} \int_{-L/2}^{L/2} e^{+ikx} f(x)$$

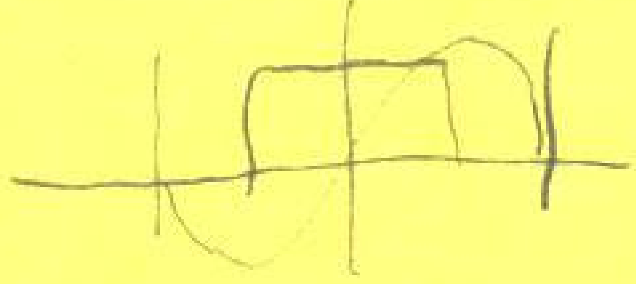
$$A_{(k)} = C(k) + C_{-k} = \frac{1}{L} \int_{-L}^{L} (e^{-ik\cdot x} + e^{+ikx})\, f(x)$$

$$\boxed{\begin{aligned} A(k) &= \frac{2}{L} \int_{-L/2}^{L/2} \cos kx\, f(x) \\ B_k &= \frac{2}{L} \int_{L/2}^{L/2} \sin kx\, f(x) \\ C_0 &= \frac{1}{L} \int_L f(x)\, dx \end{aligned}}$$

$$\boxed{A_k = \frac{1}{L}\,\frac{2\sin\frac{ka}{2}}{\frac{ka}{2}} \qquad k = \frac{2\pi n}{L} \qquad n = 0\ldots\infty}$$

$$B_k = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,\sin\frac{2\pi n}{L}x = 0$$

To see how well this works for a given # of terms see links on web



$$c_k = \frac{A_k - iB_k}{2} = \frac{1}{L}\,\frac{\sin ka/2}{ka/2} \qquad k>0$$

$$c_{-|k|} = \frac{A_{|k|} + iB_{|k|}}{2} = \frac{1}{L}\,\frac{\sin ka}{ka/2} \qquad k>0$$

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