Electrostatics in Media

$$\nabla \cdot E = \rho$$

$$\nabla \times B = j/c + 1/c\, \partial_t E$$

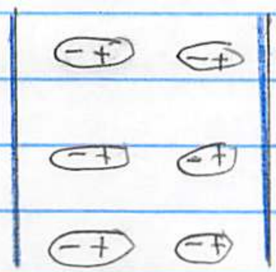
??

- But what is $\vec{j}$ in media. Need to specify a constitutive equation. Symmetry is key

$$\nabla \cdot B = 0$$

$$-\nabla \times E = 1/c\, \partial_t B$$

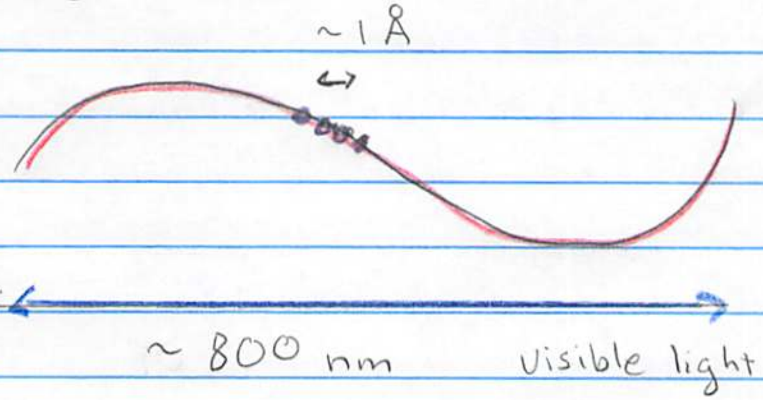
Basic Picture of Insulating Material



Electric field weakly polarizes the material

Key Points

① The wavelength of the external field is vastly longer than microscopic scales



(2) Fields are very weak compared to micro-fields

$$E_{external} \sim \frac{1\,kV}{cm} \sim 10^5 \frac{V}{m} \quad \text{(Large man made field)}$$

$$E_{atom} \sim \frac{13.6\,\cancel{e}V}{\cancel{e}\,a_0} \sim 13 \times 10^{10} \frac{V}{m}$$

$a_0 \equiv$ bohr radius $= 0.5\,\mathring{A}$

- Now we will use these points and symmetry to detrermine the medium currents

Transformation of Mechanical Quantities

Parity

$$\vec{x} \longrightarrow \underline{\vec{x}}(t) = -\vec{x}(t)$$

$$\vec{p} \longrightarrow \underline{\vec{p}}(t) = -\vec{p}(t)$$

$$\vec{L} \longrightarrow \underline{L} = +\vec{L} \qquad \vec{L} = \vec{r} \times \vec{p}$$

- Parity is a symmetry if this transformation maps a solution $x(t), p(t)$ to a new solution $\underline{x}(t), \underline{p}(t)$. Thus if parity is a symmetry

$$\frac{d\underline{p}}{dt} = F(t, \underline{x}, \underline{p})$$

$$-\frac{dp}{dt} = F(t, -x, -p)$$

- So since $dp/dt = F(t, x, p)$ we require (if we want parity as a symmetry) that

$$F(t, -x, -p) = -F(t, x, p)$$

i.e. the Forces are odd under parity, e.g.

$$\vec{F}(\vec{x}_1, \vec{x}_2) = G \frac{\vec{x}_1 - \vec{x}_2}{4\pi |\vec{x}_1 - \vec{x}_2|^3}$$ yields parity symmetric EoM

Time Reversal

- Time reversal relates a solution $x(t)$ to a new reversed solution

$$x(t) \longrightarrow \underline{x}(t) = x(-t)$$

$$p(t) \longrightarrow \underline{p}(t) = -p(-t)$$

This should read $\underline{p}(t) = -p(-t)$. Then $\dot{\underline{p}}(t) = -\frac{d}{dt}(p(-t)) = +\dot{p}(-t)$

$$L(t) \longrightarrow \underline{L}(t) = -L(t)$$

- If T-reversal is a symmetry then if $x(t), p(t)$ is a solution to the EOM obeying $dp/dt = F(t,x,p)$ for all times, then so is $\underline{x}(t)$, $\underline{p}(t)$

$$\frac{d\underline{p}}{dt} = F(t, \underline{x}, \underline{p})$$

$$+\frac{dp(-t)}{d(-t)} = F(t, x(-t), -p(-t))$$

t is just a parameter

$$\frac{dp(t)}{dt} = F(-t, x(t), -p(t))$$

- So For T-reversal symmetry require

$$F(-t, x, -p) = F(t, x, p)$$

i.e. the forces should be even under time reversal

- For example:

Take a drag force:

$$\frac{dp}{dt} = F_D(p) \qquad F_D = -\eta p$$

Under time reversal the drag force is odd

$$F_D(-p) = +\eta p = -F_D(p)$$

Thus the classical dynamis is not symmetric under time reversal with such a dissipative force

- With a potential Force $\nabla_x U(\vec{x})$ the forces are T-even and T-reversal is a good symmetry.

Thus the dynamics with this drag force is not T-reversal invariant. With a potential force $\vec{F} = \nabla_r U(\vec{r})$ the dynamics is T-reversal invariant

EM and Parity

- First consider the parity symmetric microscopic theory where the charge is a scalar under inversion

$$Q \rightarrow \underline{Q} = Q$$

- The charge density under inversion is replaced with a new parity relfected version

$$\rho(t,x) \rightarrow \underline{\rho}(t,x) = \rho(t,-x)$$

- Similarly the current density under inversion is

$$\vec{j}(t,x) \rightarrow \underline{J}(t,x) = -\vec{j}(t,-x)$$

- Then we look at the maxwell equations, and try to find a new solution $(\underline{E}, \underline{B})$ with the inverted currents and charges:

$$\nabla \cdot \underline{E}(t,x) = \underline{\rho}(t,x)$$

★
$$\nabla \times \underline{B}(t,x) = \frac{\underline{J}}{c}(t,x) + \frac{1}{c}\partial_t \underline{E}$$

$$\nabla \cdot \underline{B}(t,x) = 0$$

$$-\nabla \times \underline{E} + \frac{1}{c}\partial_t \underline{B} = 0$$

Then inspection of the equations shows that

$$\underline{E}(t,x) = -\vec{E}(t,-\vec{x})$$

$$\underline{B}(t,x) = +B(t,\vec{x})$$

i.e. $\vec{E}$ is a vector while $\vec{B}$ is a pseudo-vector. E.g.

$$\left.\begin{aligned} \nabla_{\vec{x}} \times \underline{B} &= -\nabla_{-\vec{x}} \times B(t,-x) \\ \underline{J} &= -J(t,-x) \\ \partial_t \underline{\vec{E}} &= -\partial_t E(t,-x) \end{aligned}\right\}$$

So ★ is satisfied since:

$$\nabla_{\vec{x}} \times B(t,\vec{x}) = \frac{1}{c}\vec{j}(t,x) + \frac{1}{c}\partial_t E(t,x)$$

or

$$\nabla_{-\vec{x}} \times B(t,-\vec{x}) = \frac{1}{c}\vec{j}(t,-\vec{x}) + \frac{1}{c}\partial_t E(t,-\vec{x})$$

EM and Time Reversal

- Next consider a time reversal symmetric micro-scopic theory where the charge is again a scaler $Q \to \underline{Q}$. The charge density and current under time reveral, are replaced with a new charge density and current

$$\rho(t,\vec{x}) \longrightarrow \underline{\rho}(t,\vec{x}) = \rho(-t,x)$$

$$\vec{j}(t,\vec{x}) \longrightarrow \underline{j}(t,\vec{x}) = -\vec{j}(-t,x)$$

- Then looking for a new solution $(\underline{E}, \underline{B})$ with the time reversed currents and charges:

$$\nabla \cdot \underline{E} = \underline{\rho}$$

$$\nabla \times \underline{B} = \frac{\underline{j}(t,x)}{c} + \frac{1}{c}\frac{\partial}{\partial t}\underline{E}(t,x)$$

$$\nabla \cdot \underline{B} = 0$$

★ $$-\nabla \times \underline{E}(t,x) = \frac{1}{c}\frac{\partial \underline{B}}{\partial t}(t,x)$$

So, we can determine $\underline{E}$ and $\underline{B}$ from E, B

$$\left.\begin{aligned} \underline{E}(t,x) &= +E(-t,\vec{x}) \\ \underline{B}(t,x) &= -\vec{B}(-t,x) \end{aligned}\right\}$$

i.e. $\vec{E}$ is Even under T-reversal, B is odd

E.g.

$$\left.\begin{aligned} \nabla \times \underline{E}(t,x) &= \nabla \times E(-t,x) \\ \frac{1}{c}\frac{\partial}{\partial t}\underline{B}(t,x) &= -\,-\frac{1}{c}\frac{\partial}{\partial(-t)}B(-t,x) \end{aligned}\right\}$$

So ★ follows from:

$$\nabla \times E(t,x) + \frac{1}{c}\frac{\partial B(t,x)}{\partial t} = 0$$

Summary

For a parity and T-reversal invariant electrodynamics have

	Parity	T-reversal
$x(t)$	odd	even
$p(t)$	odd	odd
$r \times p$	even	odd
F	odd	even
ρ	even	even
j	odd	odd
$\vec{E}$	Odd	Even
$\vec{B}$	Even	Odd