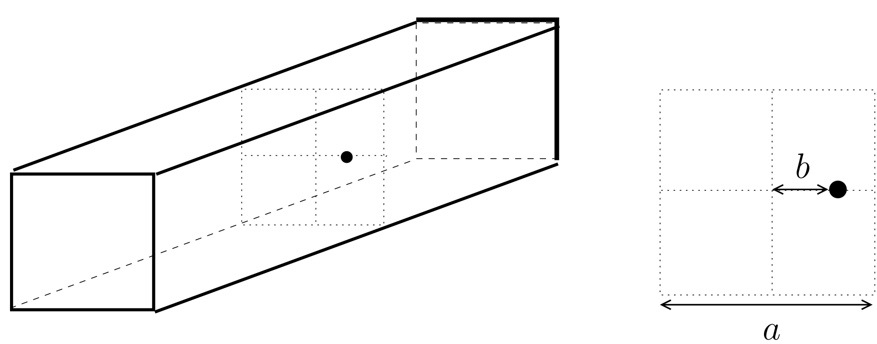
Contents

1 Electrostatics

1.1 A charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross subsectional area of the tube have length a.



1. (2 points) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x z)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \begin{cases}\cos(u) \text{ or } \sin(u)\end{cases} \tag{1}$$

for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

2. (4 points) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential.

3. (7 points) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (a) to determine the potential from the point charge described in part (b).

4. (7 points) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$.

Solution

1. The Laplace equation is

$$-\nabla^2\varphi = 0 \tag{2}$$

Separating variables with $\varphi = X(x)Y(y)Z(z)$ we must have

$$-\frac{d^2X}{dx^2} = k_x^2 X \tag{3a}$$
$$-\frac{d^2Y}{dy^2} = k_y^2 Y \tag{3b}$$
$$-\frac{d^2Z}{dz^2} = k_z^2 Z \tag{3c}$$

In order to satisfy Eq. (2), the separation constants satisfy

$$k_x^2 + k_y^2 + k_z^2 = 0 \tag{4}$$

and thus

$$\frac{d^2Z}{dz^2} = \kappa^2 Z \qquad \text{with} \qquad \kappa = \sqrt{k_x^2 + k_y^2} \tag{5}$$

The solutions to Eq. (3a) may be either sin or cos

$$X(x) = \Phi(k_x x)\,, \tag{6}$$

with k_x at this point still arbitrary. In order to satisfy the boundary conditions $X(\pm a/2) = 0$, we require for the cos functions that

$$k_x a/2 = (n + \tfrac{1}{2})\pi\,. \tag{7}$$

Similarly, for the sin functions

$$k_x a/2 = n\pi\,. \tag{8}$$

Thus, the general form is

$$X_n(x) = \Phi_n(k_n x) \qquad n = 0, 1, \ldots \tag{9}$$

with $k_n = (n+1)\pi/a$ and

$$\Phi_n(u) = \begin{cases} \cos(u) & \text{n even} \\ \sin(u) & \text{n odd} \end{cases}\,. \tag{10}$$

The $Y(y)$ direction follows by analogy

$$Y_m(y) \qquad \Phi_m(k_m x) \qquad m = 0, 1, \ldots \tag{11}$$

with $k_m = (m+1)\pi/a$ The solutions to the z direction are

$$Z(z) = e^{\pm\kappa z} \qquad \kappa = \sqrt{k_n^2 + k_m^2} \tag{12}$$

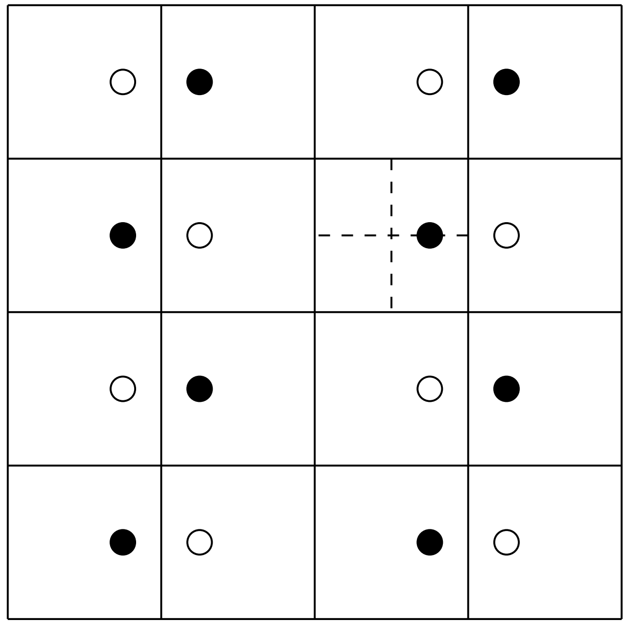


Figure 1: Arrangement of image charges. The black charges idicate plus charges, while the white charges indicate negative charges. The origin of coordinates is indicated with the dashed lines. The real charge is displaced by a distance b from the origin.

2. The image charges may be placed in a rectangular lattice as shown below. Their are four types of charges with coordinates

$$\boldsymbol{r}_1(n,m) = (b+2na)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{13}$$

$$\boldsymbol{r}_2(n,m) = ((2n+1)a-b)\hat{\boldsymbol{x}} + 2ma\hat{\boldsymbol{y}} \tag{14}$$

$$\boldsymbol{r}_3(n,m) = (b+2na)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{15}$$

$$\boldsymbol{r}_4(n,m) = ((2n+1)a-b)\hat{\boldsymbol{x}} + (2m+1)a\hat{\boldsymbol{y}} \tag{16}$$

   where n, m are integers. Then the potential is

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi}\sum_{n,m=0}^{\infty} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_1(n,m)|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_2(n,m)|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_3(n,m)|} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_4(n,m)|} \tag{17}$$

3. For the potential at $\boldsymbol{r}$ due to a point charge at $\boldsymbol{r}_o = (b,0,0)$, we expand the potential as

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) = \left(\frac{2}{a}\right)^2 \sum_{n,m=0}^{\infty,\infty} X_n(x)X_n(b)\,Y_m(y)Y_m(0)\,g_{n,m}(z) \tag{18}$$

   and substitute into the Poisson equation

$$-\nabla^2\varphi(\boldsymbol{r};\boldsymbol{r}_o) = q\delta(x-b)\delta(y)\delta(z)\,. \tag{19}$$

The leading factors $(2/a)^2$ arise from the fact that we have not normalized the eigenfunctions X and Y

$$\int_{-a/2}^{a/2} dx\, X_n(x) X_{n'}(x) = \frac{a}{2}\,\delta_{n,n'} \tag{20}$$

$$\int_{-a/2}^{a/2} dy\, Y_m(y) Y_{m'}(y) = \frac{a}{2}\,\delta_{m,m'} \tag{21}$$

If $g_{n,m}(z)$ satisfies

$$\left(k_n^2 + k_m^2 - \frac{\partial^2}{\partial z^2} \right) g_{n,m}(z) = q\delta(z)\,, \tag{22}$$

then using the completeness relation

$$\frac{2}{a} \sum_n X_n(x) X_n(x_o) = \delta(x - x_o) \tag{23}$$

$$\frac{2}{a} \sum_m Y_m(x) Y_m(x_o) = \delta(y - y_o) \tag{24}$$

it is not difficult to show that Eq. (19) is satisfied.

The solution to Eq. (22) is

$$g_{n,m}(z) = \begin{cases} Ae^{-\kappa_{n,m} z} & z > 0 \\ Ae^{\kappa_{n,m} z} & z < 0 \end{cases} \tag{25}$$

Integrating across the δ-fcn in Eq. (22) we have

$$- \left.\frac{dg}{dz}\right|_{z=0^+} + \left.\frac{dg}{dz}\right|_{0^-} = q \tag{26}$$

With this requirement $A = \frac{q}{2\kappa_{n,m}}$ and

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) = \frac{4q}{a^2} \sum_{n,m=0}^{\infty,\infty} X_n(x) X_n(b)\, Y_m(y) Y_m(0) \frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{27}$$

4. At asymptotic distances the terms with the smallest $\kappa_{n,m}$ dominate the sum. We then have only the contribution from $n = m = 0$ mode, and

$$\kappa_{0,0} = \sqrt{2}\pi/a\,. \tag{28}$$

The potential reads

$$\phi(\boldsymbol{r}; \boldsymbol{r}_o) \simeq \frac{4q}{a^2} \cos(\pi x/a) \cos(\pi b/a)\, \cos(\pi y/a) \frac{e^{-\kappa_{0,0}|z|}}{2\kappa_{0,0}} \tag{29}$$

or

$$\phi(\boldsymbol{r};\boldsymbol{r}_o) \simeq \frac{\sqrt{2}q}{\pi a} cos(\pi x/a)\cos(\pi b/a)\ \cos(\pi y/a) e^{-\sqrt{2}\pi|z|/a} \tag{30}$$

Let us calculate the charge density on the bottom plate

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{E} = -\partial_y\phi\big|_{y=-a/2}\ , \tag{31}$$

$$= -\frac{\sqrt{2}q}{a^2}\cos(\pi x/a)\cos(\pi b/a)\, e^{-\sqrt{2}\pi|z|/a}\,. \tag{32}$$

Finally, the force per area on the bottom plate is

$$\frac{F^y}{A} = \frac{\sigma^2}{2}\ , \tag{33}$$

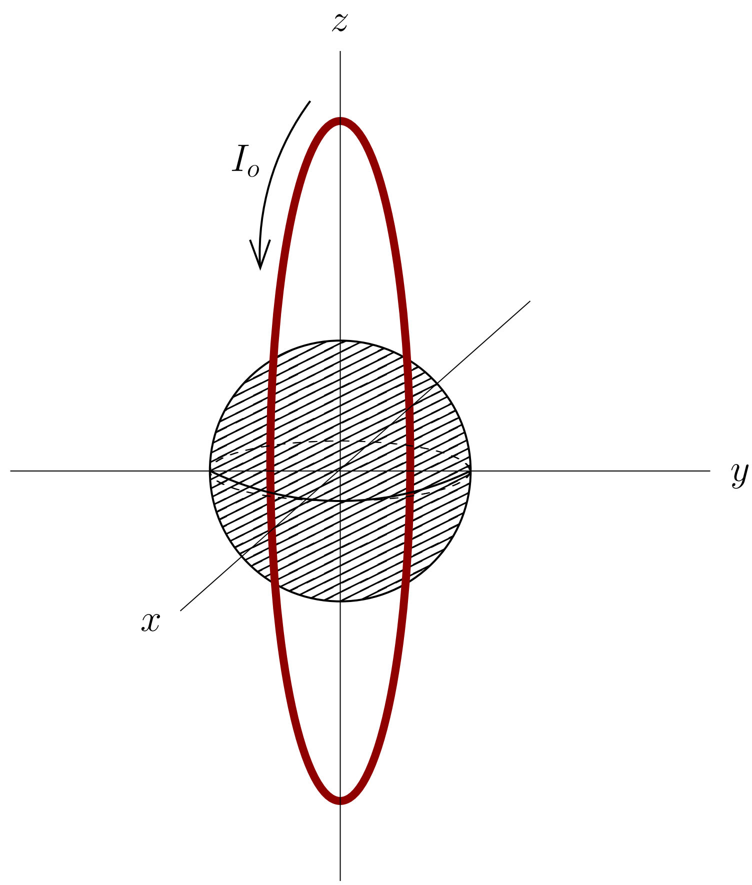
$$= \frac{q^2}{a^4}\cos^2(\pi x/a)\cos^2(\pi b/a)\, e^{-2\sqrt{2}\pi|z|/a}\,. \tag{34}$$

The direction of the force is into the tube. The other walls of the tube have the same force per area.

2 Magnetostatics

2.1 A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



1. Determine the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere.
2. Determine the bound surface current on the surface of the sphere.
3. What is the direction of the net-torque on the circular hoop? Indicate on the figure how the circular hoop will tend to rotate and explain your result.
4. Compute the net-torque on the circular hoop.

Solution

1. The magnetic field outside is one of a magnetic dipole, where all of magnetic moment is placed at the origin
$$\boldsymbol{B} = \frac{1}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \mathbf{m}\right] \tag{35}$$
Inside sphere, the magnetic field is constant
$$\boldsymbol{B} = B_o\,\hat{\boldsymbol{z}} \tag{36}$$
The constant B_o can be picked off from the boundary conditions.
The boundary conditions read
$$\boldsymbol{n}\times(\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}_b}{c} \tag{37}$$
$$\boldsymbol{n}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{38}$$
Then from the boundary conditions at $r = a$
$$B_r|_{\text{out}} = B_r|_{\text{in}}\ . \tag{39}$$
With the magnetic field outside the sphere
$$B_r|_{\text{out}} = \frac{1}{4\pi r^3}2m\cos\theta\,, \tag{40}$$
and inside the sphere
$$\hat{\boldsymbol{r}}\cdot\boldsymbol{B}|_{\text{in}} = B_o\,\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} = B_o\cos\theta\,, \tag{41}$$
comparison at $r = a$ gvies
$$B_o = \frac{1}{4\pi a^3}2m\,. \tag{42}$$
For later reference we note that with $M = m/(4\pi a^3/3)$
$$H_o = B_o - M = -\frac{m}{4\pi a^3} \tag{43}$$

2. The surface current is in the azimuthal direction
$$\boldsymbol{K} = K_o\,\hat{\boldsymbol{\phi}} \tag{44}$$
Inside we have
$$\boldsymbol{B} = B_o\hat{\boldsymbol{z}} = B_o\cos\theta\,\hat{\boldsymbol{r}} - B_o\sin\theta\,\hat{\boldsymbol{\theta}}\,, \tag{45}$$
while outside we have
$$\boldsymbol{B} = \frac{1}{4\pi r^3}2m\cos\theta\,\hat{\boldsymbol{r}} + \frac{1}{4\pi r^3}m\sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{46}$$
Then the jump condition reads
$$B_{\theta,\text{out}} - B_{\theta,\text{in}} = \frac{K_o}{c}\,. \tag{47}$$

Thus

$$K_o = c\left(\frac{1}{4\pi a^3}m + B_o\right)\sin\theta = \frac{3c}{4\pi a^3}m\sin\theta \tag{48}$$

One can verify using eq. (43)

$$H_{\theta,\text{out}} - H_{\theta,\text{in}} = \left(\frac{1}{4\pi r^3}m\sin\theta + H_o\sin\theta\right) = 0 \tag{49}$$

as should be the case since H is continuous in the absence of external macroscopic currents.

3. To compute the torque we first compute the lorentz force on a element of length $d\ell = bd\theta$.

$$\begin{aligned} dF =& \frac{I_o}{c} d\ell B_\perp & (50) \\ =& \frac{I_o}{c} bd\theta\ B_r & (51) \\ =& \frac{I_o}{c} bd\theta\ \frac{2m\cos\theta}{4\pi b^3} & (52) \end{aligned}$$

The right hand rule indicates that the force is in the $-\hat{\boldsymbol{y}}$ direction in the upper hemisphere, and in the positive $\hat{\boldsymbol{y}}$ direction in the lower hemisphere. This implies that the net torque points along the x-axis. This can be intuited by noting that the magnetic moment of the hoop tends to align with the magnetic field from the sphere

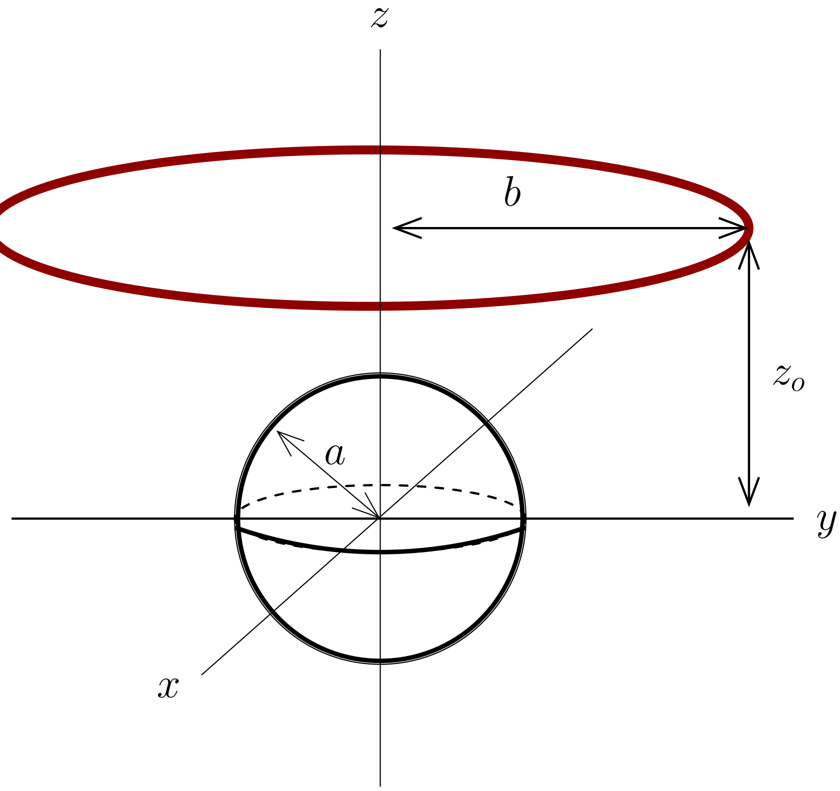
4. The torque around the x-axis

$$\begin{aligned} \tau =& \int d\tau = \int b\cos\theta\, dF & (53) \\ =& 2\int_0^\pi b\cos\theta\ \frac{I_o}{c} bd\theta\ \frac{2m\cos\theta}{4\pi b^3} & (54) \\ =& \frac{4m(I_o/c)b^2}{4\pi b^3}\int_0^\pi d\theta\cos^2\theta & (55) \\ =& \frac{4m(I_o/c)b^2}{4\pi b^3}\frac{\pi}{2} & (56) \\ =& \frac{2m}{4\pi b^3}\ \left[\frac{I_o}{c}\pi b^2\right] & (57) \end{aligned}$$

3 Quasi statics

3.1 A ring and a sphere in a magnetic field

A sphere of radius a with magnetic permeability μ is placed in an external slowly varying (homogeneous) magnetic field, $\boldsymbol{B}_{\text{ext}}(t) = B_o(t)\,\hat{\boldsymbol{z}} = \mathcal{B}\cos(\omega t)\hat{\boldsymbol{z}}$. Placed above the sphere at height z_o is an ohmic ring of radius b and resistance $\mathcal{R}$. The center of the ring coincides with the z-axis and the plane of the ring points along the z-axis (see below).



(a) (6 points) The induced magnetic moment of the sphere is proporitonal to the external field

$$\boldsymbol{m} = \alpha_B \boldsymbol{B}_{\text{ext}}\,. \tag{58}$$

Determine the polarizability, α_B. Neglect the fields from the currents induced in the ring.

(Hint: recall that for a permeable sphere in a constant external magnetic field, the magnetic field outside the sphere is that of an induced magnetic dipole plus the external field, while the magnetic field inside the sphere is constant, $\boldsymbol{B}_{\text{in}} = B_{\text{in}}\,\hat{\boldsymbol{z}}$. Determine α_B and B_{in} from the appropriate boundary conditions at the surface of the sphere.)

(b) (6 points) Determine the current induced in the ring.

(c) (2 points) Under what conditions can the induced magnetic fields from the ring be neglected in part (a)?

(d) (6 points) Determine the force on the ring.

Solution

(a) The boundary conditions read

$$\boldsymbol{n} \times (\boldsymbol{H}_{\text{out}} - \boldsymbol{H}_{\text{in}}) = 0 \tag{59}$$
$$\boldsymbol{n} \cdot (\boldsymbol{B}_{\text{out}} - \boldsymbol{B}_{\text{in}}) = 0 \tag{60}$$

In terms of components

$$H_{\theta,\text{out}} - H_{\theta,\text{in}} = 0 \tag{61}$$
$$B_{r,\text{out}} - H_{r,\text{in}} = 0 \tag{62}$$

With the magnetic field of a dipole

$$\boldsymbol{B}_{\text{out}} = B_o \, \hat{\boldsymbol{z}} + \frac{3\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^2} \tag{63}$$
$$\boldsymbol{B}_{\text{in}} = B_o \, \hat{\boldsymbol{z}} + \frac{3\hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^2} \tag{64}$$

we see that

$$B_{r,\text{out}} = \frac{2m\cos\theta}{4\pi a^3} + B_o \cos\theta \tag{65}$$
$$H_{\theta,\text{out}} = \frac{m\sin\theta}{4\pi a^3} - B_o \sin\theta \tag{66}$$

Inside we have

$$B_{r\,\text{in}} = B_{\text{in}} \cos\theta \tag{67}$$
$$H_{\theta\,\text{in}} = -\frac{1}{\mu} B_{\text{in}} \sin\theta \tag{68}$$

Putting together the ingredients we have

$$\frac{m}{4\pi a^3} - B_o + \frac{B_{\text{in}}}{\mu} = 0 \tag{69}$$
$$\frac{2m}{4\pi a^3} + B_o - B_{\text{in}} = 0 \tag{70}$$

Solving these equation for m and B_{in} we get

$$m = B_o(4\pi a^3)\frac{\mu - 1}{2 + \mu} \tag{71}$$
$$B_{\text{in}} = B_o \frac{3\mu}{2 + \mu} \tag{72}$$

(b) The flux through the loop has two contributions: the external magnetic field and the induced dipole. The external dipole contribution is simply

$$\Phi_{B,\text{ext}} = B_o(t)\pi b^2 \,. \tag{73}$$

The dipole contribution is most easily found using the vector potential

$$\Phi_{B,\rm dip} = \int \boldsymbol{B}\cdot d\boldsymbol{a} = \oint \boldsymbol{A}\cdot d\boldsymbol{\ell}\,. \tag{74}$$

With the vector potential of the dipole

$$\boldsymbol{A} = \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{4\pi r^2} \tag{75}$$

we have

$$A_\phi = \frac{m\sin\theta}{4\pi(z^2+b^2)} \tag{76}$$

So with $\sin\theta = b/\sqrt{z^2+b^2}$ we have

$$\Phi_{B,\rm dip} = \frac{m(t)}{2}\frac{b^2}{(z^2+b^2)^{3/2}} \tag{77}$$

$$= \alpha_B\frac{B_o(t)}{2}\frac{b^2}{(z^2+b^2)^{3/2}} \tag{78}$$

Thus the magnetic current is

$$I(t) = -\frac{1}{c\mathcal{R}}\partial_t\Phi_B(t) \tag{79}$$

Or

$$I(t) = \frac{-\dot{B}_o(t)\pi b^2}{c\mathcal{R}}\left[1+\frac{\alpha_B}{2\pi}\frac{1}{(z^2+b^2)^{3/2}}\right] \tag{80}$$

(c) The current in the loop produces a field at the sphere of order $I(t)/[c(b^2+z^2)^{1/2}]$. We should compare this field to B_o, yielding the condition:

$$\frac{\omega B_o\pi b^2}{c^2\mathcal{R}}\frac{1}{(z^2+b^2)^{1/2}} \ll B_o\,. \tag{81}$$

Taking b and z the same order of magnitude $b\sim z$ as drawn in the figure,

$$\frac{\omega\pi b}{2\pi c^2\mathcal{R}} \ll 1\,. \tag{82}$$

This is the answer.

It is useful to interpret the answer. The resistance is $\mathcal{R} = 2\pi b/(\sigma A)$ where A is the cross subsection of the wire and σ is the conductivity, yielding

$$\frac{\omega\sigma}{4\pi c^2}A \ll 1\,. \tag{83}$$

Recognizing the magnetic diffusion coefficient $D{=}c^2/\sigma$ of the wire and the skin depth $\delta(\omega)\sim\sqrt{D/\omega}$, we rewrite the condition as

$$\frac{A}{\pi\delta^2(\omega)} \ll 1\,. \tag{84}$$

Thus the skin depth δ should be much longer than the diameter (or thickness) of the wire.

(d) For the force we have the contribution of the constant field B_o and the field of the sphere $B_{\rm dip}$.

Using the right hand rule we see that the constant field produces no net force. All the forces of from the static field lie in the plane of the loop, tending to deform the ring but providing no net force.

From the dipole we have the Lorentz force

$$F^z = \int bd\phi \, \frac{I(t)}{c} \; \hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \boldsymbol{B}_{\rm dip}) \, . \tag{85}$$

With the diople field,

$$\boldsymbol{B}_{\rm dip} = \frac{3\hat{\boldsymbol{r}} \cdot (\hat{\boldsymbol{r}} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \, , \tag{86}$$

the magnetic moment $m(t) = \alpha_B B_o(t)\hat{\boldsymbol{z}}$, the cross products

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}}) = \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} = -\sin\theta \, , \tag{87}$$

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{z}}) = 0 \, , \tag{88}$$

we find

$$F^z = -\int bd\phi \, I(t)/c \, \frac{3\sin\theta\cos\theta \, m(t)}{4\pi(z_o^2 + b^2)^{3/2}} \tag{89}$$

Thus

$$F^z = \left(-\frac{I(t)B_o(t)b}{c} \right) \frac{3}{4} \frac{\sin(2\theta)\alpha_B}{(z_o^2 + b^2)^{3/2}} \tag{90}$$

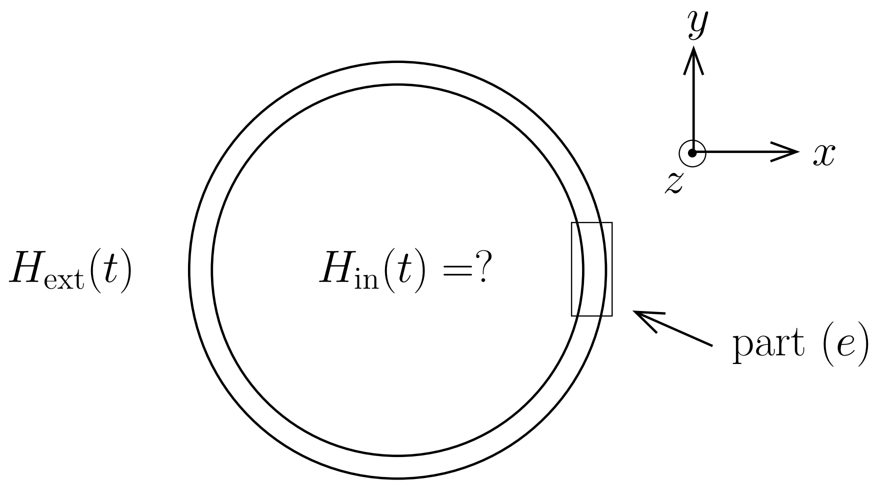
This is the answer after substituting the results of part (b).

After minor manipulations we find

$$F^z = \left(\frac{dB_o^2(t)}{dt} \frac{\pi b^3}{c^2 \mathcal{R}} \right) \frac{3}{8} \frac{\sin(2\theta)\,\alpha_B}{(z_o^2 + b^2)^{3/2}} \left[1 + \frac{\alpha_B}{2\pi(z_o^2 + b^2)^{3/2}} \right] \tag{91}$$

3.2 A cylindrical shell in a magnetic field

Consider an infinitely long cylindrical ohmic shell of conductivity σ and radius a. The walls have thickness Δ, with $\Delta \ll a$. The shell is placed in a uniform, but time dependent, external magnetic field $H_{\rm ext}(t)$, which is directed along the z-axis as shown below. The goal of this problem is to determine the magnetic field inside the cylinder. The thickness Δ is sufficiently small that the induced current density may be considered (spatially) constant inside the shell wall.



(a) (1 point) For a specified surface current $\boldsymbol{K} = K(t)\,\hat{\boldsymbol{\phi}}$, how is the magnetic field inside the shell related to the external magnetic field.

(b) (3 points) Determine a differential equation for the evolution of the magnetic field inside the cylinder. Check that your equation is dimensionfully correct.

(c) (4 points) For a sinusoidal external field, $H_{\rm ext}(t) = H_o e^{-i\omega t}$, determine the amplitude of the magnetic field's sinusoidal oscillations inside the cylinder. Make a graph of the ratio of the interior to exterior amplitudes as a function of frequency.

(d) (4 points) At higher frequency the induced current changes appreciably over the wall thickness Δ. Estimate the frequency where this (neglected) dynamics becomes important.

(e) (8 points) Determine the amplitude of magnetic field's sinusoidal oscillations inside the cylinder without assuming that the induced current is constant within the walls. Check that for small Δ you reproduce the results of part (c).

Hint: Magnify and analyze the highlighted region shown in the figure to relate the interior and exterior. Treat the walls of the cylinder as having infinite transverse (y and z) extent, so that all fields in the walls are functions x only.

Solution:

(a) First we note that for a specified current

$$\boldsymbol{n} \times (\boldsymbol{H}_{\rm ext} - \boldsymbol{H}_{\rm in}) = \frac{\boldsymbol{K}}{c} \tag{92}$$

Taking $\boldsymbol{n} = \hat{\boldsymbol{\rho}}$, $\boldsymbol{H} = H\,\hat{\boldsymbol{z}}$, $\boldsymbol{K} = K(t)\,\hat{\boldsymbol{\phi}}$, and noting that $\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{z}} = -\hat{\boldsymbol{\phi}}$ we have

$$H_{\rm ext}(t) - H_{\rm in}(t) = -\frac{K(t)}{c}\,. \tag{93}$$

One can (and should) also reason the signs in this equation using the right hand rule. Either way

$$H_{\rm in}(t) = H_{\rm ext}(t) + \frac{K(t)}{c}\,. \tag{94}$$

(b) The changing flux inside the cylinder induces a voltage. This voltage produces a current $K(t)$ given by Ohms Law. Given the current we can relate the internal and external magnetic fields through Eq. (94).

The voltage induced is

$$-\oint \boldsymbol{E}\cdot {\rm d}\boldsymbol{\ell} = \frac{1}{c}\partial_t \int \boldsymbol{B}\cdot {\rm d}\boldsymbol{a}\,. \tag{95}$$

For the geometry at hand

$$-E_\phi(2\pi a) = \frac{1}{c}\dot{H}_{\rm in}(t)\pi a^2\,, \tag{96}$$

and thus

$$E = -\frac{a}{2c}\dot{H}_{\rm in}\,. \tag{97}$$

From Ohm's Law, $\boldsymbol{J} = \sigma\boldsymbol{E}$, and the surface current $K = J\Delta$, we find

$$K = -\frac{a\Delta\sigma}{2c}\dot{H}_{\rm in}\,. \tag{98}$$

Using the boundary conditions in Eq. (94) we have finally

$$\frac{a\Delta\sigma}{2c^2}\dot{H}_{\rm in}(t) + H_{\rm in}(t) = H_{\rm ext}(t)\,. \tag{99}$$

We note that since $[\sigma] = s^{-1}$ it is easily seen that

$$\tau_m \equiv \frac{a\Delta\sigma}{2c^2}\,, \tag{100}$$

has units of time. To make sense of these numbers, note that magnetic diffusion coefficient for copper is of order

$$D \equiv \frac{c^2}{\sigma} \sim \frac{\rm cm^2}{\rm millisec}\,. \tag{101}$$

Thus the time constant of this equation is of order

$$\tau_m \sim {\rm millsec}\left(\frac{\rm cm^2}{a\Delta}\right)\,. \tag{102}$$

(c) Solving Eq. (99) for a sinusoidal steady state, $H_{\rm ext}(t) = H_{\rm o}e^{-i\omega t}$ and $H_{\rm in}(t) = H_{\rm in}e^{-i\omega t}$, we have

$$-i\omega\tau_m H_{\rm in} + H_{\rm in} = H_o\,. \tag{103}$$

Thus, $H_{\rm in} = H_o/(1 - i\omega\tau_m)$, and the oscillation amplitude is

$$|H_{\rm in}| = \frac{|H_o|}{\sqrt{1 + (\omega\tau_m)^2}}\,. \tag{104}$$

(d) At higher frequency the skin depth becomes important. The skin depth is of order

$$\delta(\omega) \sim \sqrt{\frac{D}{\omega}} \sim \sqrt{\frac{c^2}{\sigma\omega}}\,. \tag{105}$$

The dynamics changes when the skin depth is comparable to Δ

$$\delta(\omega) \sim \Delta\,. \tag{106}$$

Solving for ω, we find that the dynamics changes when

$$\omega \sim \frac{c^2}{\sigma\Delta^2}\,. \tag{107}$$

So, for a magnetic diffusion coefficient of order Eq. (101), we find

$$\omega \sim \mathrm{kHz}\left(\frac{\mathrm{cm}^2}{\Delta^2}\right)\,. \tag{108}$$

(e) Now we solve more precisely for the fields inside the walls. The magnetic fields obey the diffusion equation. This follows from Ampere's Law

$$\nabla \times \boldsymbol{B} = \frac{\sigma}{c}\boldsymbol{E}\,, \tag{109}$$

and Faraday's Law

$$-\nabla \times \boldsymbol{E} = \frac{1}{c}\partial_t \boldsymbol{B}\,. \tag{110}$$

Indeed, taking the curl of Ampere's Law, using $\nabla\times\nabla\times\boldsymbol{B} = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B}$ and $\nabla\cdot\boldsymbol{B} = 0$, we find the magnetic diffusion equation

$$D\nabla^2\boldsymbol{B} = \partial_t\boldsymbol{B}\,, \qquad D \equiv \frac{c^2}{\sigma}\,. \tag{111}$$

Since the wall thickness is very small compared to the radius, $\Delta \ll a$, we can approximate the geometry as one dimensional, up to correction of order Δ/a. the radial coordinate is in the x direction, and the ϕ direction (the direction of the electric field and current) is in the y direction. We choose $x = 0$ to be the inside wall of the cylindrical shell, so that

$x = \Delta$ is the outside wall of the cylindrical shell. The diffusion equation for sinusoidal field, $\boldsymbol{B}(t, \boldsymbol{x}) = B(x)e^{-i\omega t}\,\hat{\boldsymbol{z}}$, reads

$$\partial_x^2 B(x) = -i\frac{\omega}{D}B(x)\,. \tag{112}$$

The electric field is determined from Eq. (109)

$$-\frac{c}{\sigma}\frac{dB}{dx}(x) = E_y(x)\,. \tag{113}$$

Solving Eq. (112) we have

$$B(x) = C_0 e^{i\kappa x} + C_1 e^{-i\kappa x} \tag{114}$$

where

$$\kappa = \sqrt{\frac{i\omega}{D}} = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}. \tag{115}$$

Equivalently, we will use

$$B(x) = C_0\cos(\kappa x) + C_1\sin(\kappa x)\,, \tag{116}$$

since is slightly simpler to analyze the boundary conditions in this form.

We have boundary conditions at $x = 0$

$$B(0) = H_{\rm in} \tag{117}$$

and this sets $C_0 = H_{\rm in}$. The electric field at the $x = 0$ boundary is given by Eq. (97)

$$E_y(0) = \frac{+i\omega a}{2c}H_{\rm in}\,, \tag{118}$$

and this (through Eq. (113)) sets the derivative of $B(x)$ at $x = 0$

$$B'(0) = -\frac{\kappa^2 a}{2}H_{\rm in}\,, \tag{119}$$

fixing the coefficient $C_1 = (\kappa a)/2$. To summarize

$$B(x) = H_{\rm in}\left[\cos(\kappa x) - \frac{\kappa a}{2}\sin(\kappa x)\right]\,. \tag{120}$$

Finally since $B(\Delta) = H_{\rm ext}$ we find

$$H_{\rm in} = \frac{H_{\rm ext}}{\cos(\kappa\Delta) - \frac{\kappa a}{2}\sin(\kappa\Delta)}\,, \tag{121}$$

and thus the amplitude is

$$H_{\rm in} = \frac{H_{\rm ext}}{|\cos(\kappa\Delta) - \frac{a}{2\Delta}\kappa\Delta\sin(\kappa\Delta)|}\,. \tag{122}$$

We can check that when $\kappa\Delta \ll 1$

$$|H_{\rm in}| \simeq \frac{|H_{\rm ext}|}{|1 - i\omega\tau_m|} = \frac{H_{\rm ext}}{\sqrt{1 + (\omega\tau_m)^2}}\,, \tag{123}$$

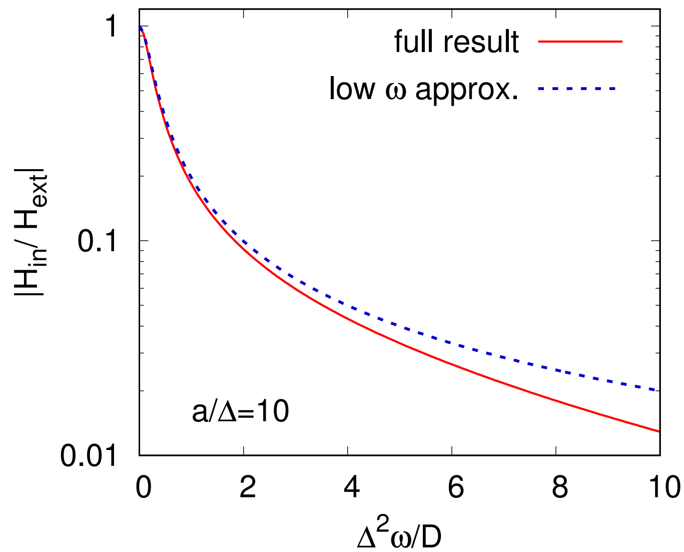


Figure 2: The field in the center divided by the external field. See text for further explanation

where we have recognized that

$$\frac{\kappa^2 a\Delta}{2} = i\omega\tau_m \,. \tag{124}$$

We have assumed that $a/\Delta \gg 1$. Thus for $(\kappa\Delta)^2 \gg \Delta/a$ we can neglect the $\cos(\kappa\Delta)$ term in comparison to the $\sin(\kappa\Delta)$ term in the denominator of Eq. (122). For $(\kappa\Delta)^2 \sim \Delta/a$ we may approximate $\cos(\kappa\Delta) \simeq 1$ up to correction of order a/Δ. Thus in a uniform approximation (i.e. an approximation which is valid for all $\kappa\Delta$) we have

$$|H_{\rm in}| = \frac{H_{\rm ext}}{|1 - i\omega\tau_m \frac{\sin(\kappa\Delta)}{\kappa\Delta}|} \,, \tag{125}$$

which is our final result.

Taking $a/\Delta = 10$ for instance, we plot the full result (Eq. (125)) and its low frequency approximation (Eq. (104)) in Fig. 2. At large frequency the skin-depth leads to exponential suppression, rather than the $1/\omega$ behaviour predicted by the low frequency approximation.

3.3 (Part of) Induction and the energy in static Magnetic fields

Consider a closed circuit of wire formed into a circular coil of n turns with radius a, resistance R, and self-inductance L. The coil rotates around the z-axis in a uniform magnetic field H directed along the x-axis (see below).

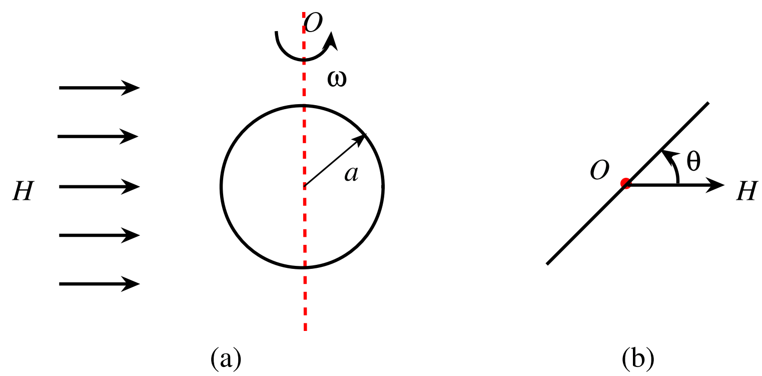


Figure 3: (a) side view; (b) top view.

a) (6 points) Find the current in the coil as a function θ for rotation at a constant angular velocity ω. Here $\theta(t) = \omega t$ is the angle between the plane of the coil and H (the x-axis).

b) (4 points) Find the externally applied torque that is needed to maintain the coil's uniform rotation.

Note: in all parts you should assume that all transient effects have died away.

Solution:

a) Let I be the current in the coil, we have

$$\oint_{\text{coil}} E \cdot dr = IR = -L\frac{dI}{dt} - \frac{1}{c}\frac{\partial \Phi_H}{\partial t}\,, \tag{126}$$

where the flux is given by $\Phi_H = \pi a^2\, n\, H \sin\theta(t)$ with $\theta(t) = \omega t$. With these phase conventions, the area vector of the loop points in the negative $\hat{\boldsymbol{y}}$ direction at $t = 0$ and in the $\hat{\boldsymbol{x}}$ direction at $\omega t = \pi/2$. Thus the circulation of a positive current at $t = 0$ is specified with the right hand rule with the thumb pointing in the negative $\hat{\boldsymbol{y}}$ direction.

From Eq. (126), we have the differential equation for the current,

$$L\,\frac{dI}{dt} + R\,I = -\frac{\pi a^2}{c} n\,H\,\omega\, \cos(\omega t)\,. \tag{127}$$

We will write this as

$$L\,\frac{dI}{dt} + R\,I = -\frac{\pi a^2}{c} n\,H\,\omega\, e^{-i\omega t}\,. \tag{128}$$

with the understanding that one is supposed to take the real part. Taking a trial solution $I(t) = I_\omega e^{-i\omega t}$, we solve for I_ω and find

$$I_\omega = \frac{\pi a^2\, n\, H\,\omega}{c}\frac{1}{R - i\omega L}\,. \tag{129}$$

Thus

$$\begin{aligned} I(t) &= -\frac{\pi a^2\, n\, H\,\omega}{c}\,\frac{1}{2}\left[\frac{e^{i\omega t}}{R + i\omega L} + \frac{e^{-i\omega t}}{R - i\omega L}\right] \\ &= -\frac{\pi a^2\, n\, H}{c}\frac{\omega}{\sqrt{R^2 + \omega^2 L^2}}\, \cos(\omega t + \phi)\,, \end{aligned} \tag{130}$$

where the phase $\phi = \tan^{-1}(-\omega L/R)$.

b) The rotating coil has a magnetic dipole moment, $\boldsymbol{\mu}(t) = I(t)\vec{A}(t)/c$. With the conventions of the previous part we have

$$\boldsymbol{\mu}(t) = m_o \cos(\omega t + \phi)\, \left(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}\right)\,. \tag{131}$$

where

$$m_o \equiv \left(\frac{\pi a^2 n}{c}\right)^2 \frac{\omega}{\sqrt{R^2 + \omega^2 L^2}} H\,. \tag{132}$$

The torque on the loop is $\boldsymbol{\mu} \times \mathbf{H}$, and an external torque of $\boldsymbol{\tau}_{\text{ext}} = -\boldsymbol{\mu} \times \mathbf{H}$ is needed to keep the coil rotating at a constant angular velocity is (with $\mathbf{H} = H\hat{\boldsymbol{x}}$) :

$$\boldsymbol{\tau}_{\text{ext}}(t) = m_o H \cos(\omega t + \phi) \cos(\omega t)\, \hat{\boldsymbol{z}} \tag{133}$$

3.4 (Part of) A time dependent dipole

Consider an electric dipole at the spatial origin ($\boldsymbol{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, *i.e.*

$$\boldsymbol{p}(t) = p_o \cos(\omega t)\hat{\boldsymbol{z}}\,, \tag{134}$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the z direction.

1. Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.

2. In the near field regime, *estimate* how the electric and magnetic field strengths decrease with the radius r. (r is the distance from the origin to the observation point.)

3. Using a system of units where $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units (such as Gaussian or Heaviside-Lorentz), *estimate* the ratio E/B at a distance r in the near field[1]. Is this ratio large or small?

4. Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.

[1] In SI units this question reads, "Estimate the ratio E/cB at a distance r in the near field."

Solution

1. The speed of light and the frequency define a length scale

$$1/(R_o) = \omega/c$$

For distances less than R_o a quasi-static approximation may be used. For distances greater than R_o the finite speed of light must be considered to calculate the radiation fields

2. There are various ways to do this. Perhaps the most direct is to use the gauge potentials in the lorentz gauge. We will not do this, but use the Maxwell equations directly.

The electric field in the near field region is just the field of a dipole

$$\boldsymbol{E} = \frac{1}{4\pi r^3}\left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}\right] \tag{135}$$

Clearly $\boldsymbol{E}$ lies in $\hat{r}, \hat{\theta}$ plane. So

$$\boldsymbol{E} = \frac{1}{4\pi r^3}\left[(2p_o(t)\cos\theta)\,\hat{\boldsymbol{r}} + (p_o(t)\sin\theta)\,\hat{\boldsymbol{\theta}}\right] \tag{136}$$

where $p_o(t) = p_o\cos(\omega t)$

Since

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\partial_t\boldsymbol{E} \tag{137}$$

We try $\boldsymbol{B}$ in the ϕ direction, with $B_\phi(r,\theta)$. Then

$$(\nabla\times\boldsymbol{B})_\theta = -\frac{1}{r}\partial_r(rB_\phi) = \frac{1}{4\pi r^3}(\partial_t p_o)\sin\theta \tag{138}$$

Integrating with respect to r we find

$$B_\phi = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta + \frac{f(\theta)/R_o^2}{r} \tag{139}$$

Where $f(\theta)$ is a dimensionless integration constant, and we have inserted factors of R_o to make up the dimensions. The terms proportional to $1/r$ can be dropped in the near field regime since it is smaller by r/R_o than the $\frac{1}{r^2}$ term. Thus

$$B_\phi = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta\,. \tag{140}$$

Then one verifies that

$$(\nabla\times\boldsymbol{B})_r = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta B_\phi) = \frac{1}{4\pi r^3 c}(\partial_t p_o)\,2\cos\theta = \frac{1}{c}\partial_t E_r \tag{141}$$

showing that B_ϕ satisfies the Maxwell equations.

Another way to do this is by recognizing a formal simlarity to the magnetic dipole. The vector potential of a magnetic dipole satisfies

$$\nabla \times \boldsymbol{A} = \boldsymbol{B} \text{ of a dipole} = \frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \tag{142}$$

and equals

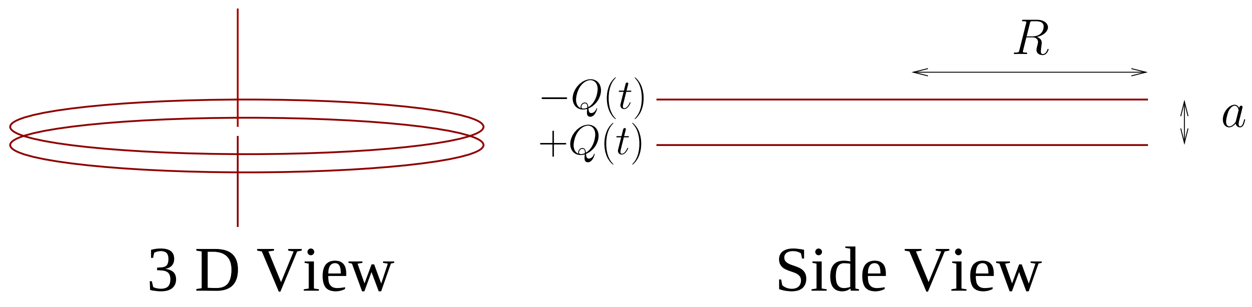
$$\boldsymbol{A} = \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{4\pi r^2}\,. \tag{143}$$

Here we are trying to solve

$$\nabla \times \boldsymbol{B} = \frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\dot{\boldsymbol{p}}(t)/c) - \dot{\boldsymbol{p}}(t)/c}{4\pi r^3}\,. \tag{144}$$

So we have (by analogy with the magnetic dipole)

$$\boldsymbol{B} = \frac{\dot{\boldsymbol{p}}(t)/c\times\hat{\boldsymbol{r}}}{4\pi r^2} = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta\,\widehat{\boldsymbol{\phi}} \tag{145}$$



3.5 A circular capicitor

A circular capacitor of radius R and separation a, with $a \ll R$, is charged with a slow sinusoidal current, *i.e.* the charge on the plates is $Q(t) = \pm Q_o \sin(\omega t)$ as illustrated above. Neglect any fringing of the fields.

1. Determine the electric and magnetic fields in between the plates in a quasi-static approximation. Draw a picture to indicate the directions of the fields while the charge on the bottom plate is positive and increasing.

2. What are the size of typical corrections to the fields computed in part (1) due to the finite speed of light?

3. Write down the Maxwell equations for the gauge potentials ϕ and $\mathbf{A}$ in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$

4. Determine the gauge potentials $(\phi, \mathbf{A})$ associated with the fields of part (1) and show that that they satisfy the Maxwell equations found in part (3) to the required order.

   The curl of a vector field $\mathbf{F}$ in cylindrical coordinates is with $\rho = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$

$$\nabla \times \mathbf{F} = \left(\frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) \hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho} \frac{\partial (\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) \hat{\mathbf{z}} \tag{146}$$

   The Laplacian is

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial^2 f}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} \tag{147}$$

Solution – Heavy Side Lorentz Units

1. The electric field is

$$\nabla \cdot \mathbf{E} = \rho_Q \qquad E^z = \frac{Q(t)}{\pi R^2}\hat{\mathbf{z}} \tag{148}$$

where ρ_Q is used to distinguish the charge density ρ_Q from the radial coordinate ρ. The magnetic field is determined from Amperes law with no current

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} - \nabla \times \mathbf{B} = 0 \tag{149}$$

So

$$B^\phi(2\pi\rho) = \frac{1}{c}\pi\rho^2\partial_t E^z \tag{150}$$

Or since $Q(t) = Q_o \sin(\omega t)$

$$B^\phi = \frac{\rho\omega}{2c}\frac{Q_o}{\pi R^2}\cos(\omega t) \tag{151}$$

2. Corrections are of order

$$\left(\frac{R\omega}{c}\right)^2 \tag{152}$$

3. Then we have

$$-\Box\varphi - \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t\phi + \nabla \cdot \boldsymbol{A}\right) = \rho_Q \tag{153}$$

$$-\Box\boldsymbol{A} + \partial_i\left(\frac{1}{c}\partial_t\phi + \nabla \cdot \boldsymbol{A}\right) = \boldsymbol{j}/c \tag{154}$$

Taking $\rho_Q = 0$ and $\boldsymbol{j} = 0$. Then taking the Coulomb gauge $\nabla \cdot A = 0$ we have

$$-\nabla^2\varphi = \rho_Q \tag{155}$$

$$-\Box\boldsymbol{A} = -\partial_i\left(\frac{1}{c}\partial_t\phi\right) + \boldsymbol{j}/c \tag{156}$$

4. Solving the ϕ

$$-\partial_i\partial^i\phi = 0 \qquad \Longrightarrow \qquad \phi = \frac{Q_o\sin(\omega t)}{\pi R^2}z \tag{157}$$

where we have implicitly assumed that the potential is not corrected beyond its zeroth order form.

For $\boldsymbol{A}$ we have only a z component. And, we may drop ∂_t^2 in the quasi static approximation

$$-\frac{1}{c^2}\partial_t^2 A^z + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A^z}{\partial\rho}\right) = \frac{1}{c}\partial_t\partial^z\phi \tag{158}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A^z}{\partial\rho}\right) = \frac{-\omega}{c}\frac{Q_o}{\pi R^2}\cos(\omega t) \tag{159}$$

Integrating this last equation we find

$$A^z = -\frac{Q_o}{\pi R^2}\cos(\omega t)\frac{\omega\rho^2}{4c} + A^z_{\rm homo}(t,\rho) \tag{160}$$

where

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A^z_{\rm homo}}{\partial\rho}\right) = 0 \tag{161}$$

is a homogeneous solution to the differenctial equation. The solutions to this equation are of the form:

$$A^z_{\rm homo}(t,\rho) = C_0(t) + C_1(t)\log(\rho)\,. \tag{162}$$

The C_1 term is irrregular at $\rho = 0$ and may be discarded. The $C_0(t)$ term is fixed by the requirement that the charge on the plates is $Q(t)$. The electric field is

$$E^z = -\nabla\phi - \frac{1}{c}\partial_t A^z\,, \tag{163}$$

and thus the charge density is

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{E} = E^z = \frac{Q_o\sin(\omega t)}{\pi R^2} - \frac{Q_o\sin(\omega t)}{\pi R^2}\left(\frac{\omega^2\rho^2}{4c^2}\right) - \frac{1}{c}\dot{C}_0(t)\,. \tag{164}$$

The integral of the charge density is fixed by the condition

$$Q_o\sin(\omega t) = \int_0^R (2\pi\rho{\rm d}\rho)\,\sigma(t,\rho)\,, \tag{165}$$

leading to the requirement that

$$C_0(t) = \frac{Q_o\cos(\omega t)}{\pi R^2}\frac{\omega R^2}{8c}\,. \tag{166}$$

The final result for A^z thus reads

$$A^z(t) = -\frac{Q_0\cos(\omega t)}{\pi R^2}\left(\frac{\omega\rho^2}{4c} - \frac{\omega R^2}{8c}\right)\,. \tag{167}$$

A straight forward sanity check gives $\mathbf{B} = \nabla\times A$

$$B^\phi = -\frac{\partial}{\partial\rho}A^z = \frac{Q_o}{\pi R^2}\frac{\rho\omega}{2c}\cos\omega t \tag{168}$$

4 Waves

4.1 Waves in Metals

Consider an ohmic metal with high (but not infinite) conductivity σ and magnetic permeability[2] $\mu = 1$, so that $\boldsymbol{B} = \boldsymbol{H}$.

1. (6 pnts) Show that for harmonic time dependence, and high conductivity[3] $\sigma \gg \omega$, that damped wave like solutions propagating in z-direction in the metal take the approximate form:

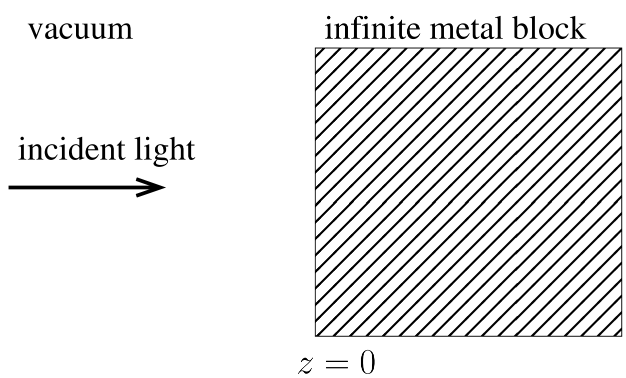
$$\boldsymbol{H}(t, z) = \boldsymbol{H}_c e^{-i\omega t + ik_c z} \tag{169}$$

where[4]

$$k_c = \frac{1+i}{\sqrt{2}} \frac{\sqrt{\sigma\omega}}{c} \tag{170}$$

2. (4 pnts) The electric field obeys a similar equation, $\boldsymbol{E}(t, z) = \boldsymbol{E}_c e^{-i\omega t + ik_c z}$. Use the Maxwell equations to express the amplitude of the electric field $\boldsymbol{E}_c$ in terms of the magnetic field $\boldsymbol{H}_c$.

3. (4 pnts) Now consider a linearly polarized plane wave in vacuum of frequency ω, which is normally incident upon a semi-infinite metal block with *infinite* conductivity as shown below.



When the metal has infinite conductivity, the amplitude of the reflected equals equals the amplitude of the incident wave, but the polarization of the reflected wave is inverted. Explain this familiar fact using the appropriate boundary conditions.

4. (6 pnts) Now consider the same reflection problem as in part 3, but this time the metal has a large (but finite) conductivity σ. Determine the electric and magnetic fields in the metal to leading order in ω/σ. The amplitude of the incident wave is E_o.

5. (not part of exam). Determine the energy lost into the metal in terms of the input magnetic field. (See lecture for two different ways to do this).

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[2] In SI units this reads $\mu = \mu_o$

[3] In SI units this condition reads $(\sigma/\epsilon_o) \gg \omega$

[4] This is written in Heaviside-Lorentz units. In SI units $k_c = (1+i)/\sqrt{2}\,\sqrt{\omega(\sigma/\epsilon_o)}\,/c$, while in Gaussian units, $k_c = (1+i)/\sqrt{2}\,\sqrt{4\pi\sigma\omega}\,/c$.

Solution

1. Writing the Maxwell Equations for harmonic fields

$$\nabla \cdot \boldsymbol{E} = \rho \tag{171}$$

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{J}}{c} - i\omega \boldsymbol{E} \tag{172}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{173}$$

$$\nabla \times \boldsymbol{E} = + i\frac{\omega}{c}\boldsymbol{B} \tag{174}$$

we then use $\boldsymbol{J} = \sigma \boldsymbol{E}$, and substitute $\boldsymbol{E} = \boldsymbol{E}_c \, e^{ik\boldsymbol{n}\cdot\boldsymbol{x}}$, with $\boldsymbol{n} = \hat{\boldsymbol{z}}$, and $\boldsymbol{H} = \boldsymbol{H}_c e^{ik\boldsymbol{n}\cdot\boldsymbol{x}}$, we have then

$$ik\boldsymbol{n} \times \boldsymbol{B}_c = \frac{\sigma}{c}\boldsymbol{E}_c - i\omega \boldsymbol{E}_c \tag{175}$$

$$ik\boldsymbol{n} \times \boldsymbol{E}_c = + \frac{i\omega}{c}\boldsymbol{B}_c \tag{176}$$

So: dropping the second term on the first equation (since $\sigma \gg \omega$); taking $ik\boldsymbol{n} \times$(the first equation); using the second equation to handle $ik\boldsymbol{n} \times \boldsymbol{E}_c$; manipulating the double cross product with the "b (ac) - (ab)c" rule; and finally using that $ik\boldsymbol{n} \cdot \boldsymbol{B} = 0$ gives

$$k^2 \boldsymbol{B}_c = \frac{i\sigma\omega \boldsymbol{B}_c}{c^2} \tag{177}$$

Or

$$k = \sqrt{\frac{i\sigma\omega}{c^2}} = e^{i\phi}\frac{\sqrt{\sigma\omega}}{c} \tag{178}$$

where $e^{i\phi} = (1+i)/\sqrt{2}$.

2. Using

$$ik\boldsymbol{n} \times \boldsymbol{E}_c = \frac{i\omega}{c}\boldsymbol{B}_c \tag{179}$$

Fom the $\nabla \cdot \boldsymbol{E} = \rho$ equation we get $\boldsymbol{n} \cdot \boldsymbol{E}_c = 0$ after writing using current convservation, $\rho = ik\boldsymbol{n} \cdot \boldsymbol{E}/(i\omega)$. Thus we make cross both sides with $\boldsymbol{n}$, use "b(ac) - (ab) c" rule to find:

$$\boldsymbol{E}_c = \frac{\omega}{ck}\left(-\boldsymbol{n} \times \boldsymbol{B}_c\right) \tag{180}$$

This says that

$$\boldsymbol{E}_c = \sqrt{\frac{\omega}{\sigma}} e^{-i\phi}\left(-\boldsymbol{n} \times \boldsymbol{B}_c\right) \tag{181}$$

3. We write the the electric field in vacuum as a sum of the incident and reflected wave

$$\boldsymbol{E}_{\text{vac}} = E_I \hat{\boldsymbol{x}} e^{ikz - i\omega t} + E_R \hat{\boldsymbol{x}} e^{-ikz - i\omega t} \tag{182}$$

$$\boldsymbol{H}_{\text{vac}} = E_I \hat{\boldsymbol{y}} e^{ikz - i\omega t} - E_R \hat{\boldsymbol{y}} e^{-ikz - i\omega t} \tag{183}$$

while inside the metal the electric fields are zero. Thus the boundary condition

$$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0\,, \tag{184}$$

yields

$$\boldsymbol{E}_{\rm vac}\big|_{z=0} = 0\,. \tag{185}$$

Or

$$E_I = -E_R \tag{186}$$

4. The boundary values of the vacuum fields are

$$\boldsymbol{E}_{\rm vac} = (E_I + E_R)\hat{\boldsymbol{x}} \tag{187}$$

$$\boldsymbol{H}_{\rm vac} = (E_I - E_R)\hat{\boldsymbol{y}} \tag{188}$$

Inside the conductors, the boundary values of the conductor fields

$$\boldsymbol{E}_c = H_c e^{-i\phi}\sqrt{\frac{\omega}{\sigma}}\hat{\boldsymbol{x}} \tag{189}$$

$$\boldsymbol{H}_c = H_c\hat{\boldsymbol{y}} \tag{190}$$

The boundary conditions

$$\boldsymbol{n} \times (\boldsymbol{E}_c - \boldsymbol{E}_{\rm vac}) = 0 \tag{191}$$

and

$$\boldsymbol{n} \times (\boldsymbol{H}_c - \boldsymbol{H}_{\rm vac}) = 0 \tag{192}$$

So

$$E_I + E_R = H_c\sqrt{\frac{\omega}{\sigma}}e^{-i\phi} \tag{193}$$

$$E_I - E_R = H_c \tag{194}$$

And solving

$$H_c \simeq 2E_I\left(1 - \sqrt{\frac{\omega}{\sigma}}e^{-i\phi}\right) \tag{195}$$

while

$$E_c \simeq 2E_I\sqrt{\frac{\omega}{\sigma}}e^{-i\phi} \tag{196}$$

5. So the energy loss per incident flux is found by evaluating the Poynting vector just inside the metal

$$\frac{\langle \boldsymbol{S}\cdot\boldsymbol{z}\rangle}{\frac{c}{2}|E_I|^2} = \frac{{\rm Re}[E_c H_c^*]}{|E_I|^2} = 4\sqrt{\frac{\omega}{\sigma}}{\rm Re}[e^{i\phi}] = 2\sqrt{2}\sqrt{\frac{\omega}{\sigma}} \tag{197}$$

4.2 Dispersion in collisionless plasmas with an external magnetic field

Model a cold non-relativistic collisionless plasma as a system of non-interacting classical electrons of uniform number density n_0. The electrons have charge q and mass m and are initially at rest. The electrons sit in a stationary and uniform background of positive charges of charge density $+|q|n_0$, whose only role in this problem is to neutralize the overall charge of the system. In the presence of an external electromagnetic field the electrons begin to move according to the classical equation of motion

$$m\frac{d^2\boldsymbol{x}}{dt^2} = q(\boldsymbol{E}(t,\boldsymbol{x}) + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}(t,\boldsymbol{x}))\,. \tag{198}$$

Consider an electromagnetic plane wave with electric field $\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{E}_0\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$ propagating in the plasma. The amplitude $\boldsymbol{E}_0$ is sufficiently small that the plasma is only weakly perturbed.

(a) (3 points) Determine the current density $\boldsymbol{j}(t,\boldsymbol{x})$ induced by the plane wave. Express your results in terms of the plasma frequency $\omega_p^2 = \frac{q^2 n_0}{m}$.

Hint: Work to leading order in the amplitude of the external field $\boldsymbol{E}_0$, so that an electron's position is constant up to small corrections proportional to $\boldsymbol{E}_0$, $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \delta\boldsymbol{x}(t,\boldsymbol{x}_0)$.

(b) (3 points) Determine the induced charge density $\rho(t,\boldsymbol{x})$. Show that $\boldsymbol{E}_0$ is transverse to $\boldsymbol{k}$ for generic frequency ω.

(c) (5 points) Determine the permittivity of the plasma, $\epsilon(\omega)$, as a function of frequency. Find a dispersion relation, $k(\omega)$, for the electromagnetic plane wave. For what range of frequencies will the plane wave propagate in the plasma? Explain.

(d) (3 points) For $\omega \gg \omega_p$, how much does the group velocity of the wave deviate from the vacuum speed of light?

Now place the plasma in a strong time independent and homogeneous magnetic field of magnitude $\mathcal{B}_0$ pointing in z direction. We will reanalyze the dispersion relation when the additional magnetic field is present. For circularly polarized waves with $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ in the z direction, the the electric field take the form

$$\boldsymbol{E}_\pm(t,\boldsymbol{x}) = E_0 \boldsymbol{\epsilon}_\pm e^{-i\omega t + ikz}, \quad \text{with} \quad \boldsymbol{\epsilon}_\pm \equiv \frac{\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}}{\sqrt{2}}. \tag{199}$$

(e) (2 points) Determine the current induced by the circularly polarized waves. Express your result in terms of the plasma frequency ω_p^2 and the cyclotron frequency[5] $\Omega_c = q\mathcal{B}_0/mc$.

Hint: Assume that $\delta\boldsymbol{x}(t,\boldsymbol{x}_0)$ is proportional to $\boldsymbol{\epsilon}_\pm$ and work to leading order in the electric field.

[5] In SI units $\Omega_c = q\mathcal{B}_0/m$.

(f) (4 points) Determine the dispersion relation $k_\pm(\omega)$ of circularly polarized plane waves in the presence of $\mathcal{B}_0$. Describe qualitatively how linearly polarized light at high frequency $\omega \gg \omega_p$ would change upon traversing a region of weak magnetic field.

Solution

(a) The electron coordinate is peturbed from its equilibrium position harmonically:

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \underbrace{\boldsymbol{x}_\omega(\boldsymbol{x}_0)e^{-i\omega t}}_{\equiv \delta\boldsymbol{x}(t,\boldsymbol{x}_0)}\,, \tag{200}$$

where here and below we notate harmonic time dependence of the variables with a subscript, *e.g.*

$$\boldsymbol{E}(t,\boldsymbol{x}_0) = \boldsymbol{E}_\omega(\boldsymbol{x}_0)e^{-i\omega t}\,, \qquad \boldsymbol{E}_\omega(\boldsymbol{x}_0) \equiv \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0}\,. \tag{201}$$

Substituting Eq. (200) into the Newtonian equations of motion and solving to first order $\boldsymbol{E}_0$ and $\delta\boldsymbol{x}$ we find

$$-m\omega^2 \boldsymbol{x}_\omega e^{-i\omega t} = q\boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0 - i\omega t}\,, \tag{202}$$

yielding

$$\boldsymbol{x}_\omega(\boldsymbol{x}_0) = -\frac{q\boldsymbol{E}_0(\boldsymbol{x}_0)}{m\omega^2}\,. \tag{203}$$

Thus the harmonic current at point $\boldsymbol{x}_0$ is

$$\boldsymbol{j}(t,\boldsymbol{x}_0) = qn_0\boldsymbol{v}(t,\boldsymbol{x}_0) = -\frac{n_0q^2}{m\omega^2}\left(-i\omega\boldsymbol{E}_\omega(\boldsymbol{x}_0)e^{-i\omega_t}\right)\,, \tag{204}$$

$$= -\frac{\omega_p^2}{\omega^2}\left(-i\omega\boldsymbol{E}_\omega(\boldsymbol{x}_0)\,e^{-i\omega t}\right)\,, \tag{205}$$

where we have defined the plasma frequency

$$\omega_p^2 \equiv \frac{n_0q^2}{m}\,. \tag{206}$$

(b) Once the current is specified the continuity equation

$$\partial_t\rho(t,\boldsymbol{x}_0) + \nabla_{\boldsymbol{x}_0}\cdot\boldsymbol{j}(t,\boldsymbol{x}_0) = 0\,, \tag{207}$$

determines the induced charge density

$$\rho_\omega(\boldsymbol{x}_0) = \frac{\boldsymbol{k}\cdot\boldsymbol{j}_\omega(\boldsymbol{x}_0)}{\omega}\,, \tag{208}$$

$$= \frac{\omega_p^2}{\omega^2}\,i\boldsymbol{k}\cdot\boldsymbol{E}_0e^{i\boldsymbol{k}\cdot\boldsymbol{x}_0}\,. \tag{209}$$

The Gaus law gives

$$\nabla\cdot\boldsymbol{E}_\omega(\boldsymbol{x}_0) = \rho_\omega(\boldsymbol{x}_0)\,, \tag{210}$$

yielding

$$i\boldsymbol{k}\cdot\boldsymbol{E}_0 = \frac{\omega_p^2}{\omega^2}\,i\boldsymbol{k}\cdot\boldsymbol{E}_0\,. \tag{211}$$

For generic frequency this equation requires that $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$, i.e. $\boldsymbol{E}_0$ is transverse. For the specific frequency $\omega = \omega_p$, longtidunal modes, known as plasma oscillations, are possible. Except at this frequency, the induced charged density is zero.

(c) The frequency dependent dielectric susceptibility is defined through the linear constitutive equation

$$\boldsymbol{j}_\omega(\boldsymbol{x}_0) = -i\omega\chi(\omega)\boldsymbol{E}_\omega(\boldsymbol{x}_0)\,, \tag{212}$$

and thus comparing Eq. (212) and Eq. (204) we find

$$\epsilon(\omega) = 1 + \chi(\omega) \qquad \chi(\omega) = -\frac{\omega_p^2}{\omega^2}\,. \tag{213}$$

In terms of χ the density reads

$$\rho = -\chi(\omega)(i\boldsymbol{k}\cdot\boldsymbol{E}_\omega) \tag{214}$$

Given the linear constitutive relations and the Maxwell equations

$$\nabla\cdot\boldsymbol{E}_\omega = \rho_\omega\,, \tag{215}$$

$$+i\frac{\omega}{c}\boldsymbol{E}_\omega + \nabla\times\boldsymbol{B}_\omega = \frac{\boldsymbol{j}_\omega}{c}\,, \tag{216}$$

$$\nabla\cdot\boldsymbol{B}_\omega = 0\,, \tag{217}$$

$$-i\frac{\omega}{c}\boldsymbol{B}_\omega + \nabla\times\boldsymbol{E}_\omega = 0\,, \tag{218}$$

we deduce that

$$\frac{\omega^2}{c^2}\epsilon(\omega) - k^2 = 0\,. \tag{219}$$

Thus, there are nontrivial solutions for specific values of k:

$$k(\omega) = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}\,. \tag{220}$$

For frequencies less than the plasma frequency, k is imaginary and the plasma does not support travelling waves. For frequencies greater than ω_p travelling waves are supported.

(d) At large frequencies we have

$$k(\omega) \simeq \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{2\omega^2}\right)\,, \tag{221}$$

and we may solve approximately for $\omega(k)$

$$\omega(k) \simeq ck\left(1 + \frac{\omega_p^2}{2(ck)^2}\right)\,. \tag{222}$$

Differentiating with respect to k we determine the group velocity

$$v_g = \frac{d\omega}{dk} \simeq c\left(1 - \frac{\omega_p^2}{2c^2k^2}\right)\,. \tag{223}$$

Notice that the phase velocity $\omega(k)/k$ is greater than the speed of light, while the group velocity is less than the speed of light as should be the case.

(e) Now we have a strong magnetic field in the z direction. Since the light is circularly polarized we try the suggested ansatz

$$\boldsymbol{x}(t, \boldsymbol{x}_0) = x_\omega(\boldsymbol{x}_0) e^{-i\omega t} \boldsymbol{\epsilon}_+ \,. \tag{224}$$

The velocity is

$$\boldsymbol{v}(t, \boldsymbol{x}_0) = -i\omega x_\omega(\boldsymbol{x}_0) e^{-i\omega} \boldsymbol{\epsilon}_+ \,, \tag{225}$$

and $\boldsymbol{v} \times \boldsymbol{B}$ is proportional to

$$\boldsymbol{\epsilon}_\pm \times \hat{\boldsymbol{z}} = (\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}) \times \hat{\boldsymbol{z}} \,, \tag{226}$$

$$= (-\hat{\boldsymbol{y}} \pm i\hat{\boldsymbol{x}}) \,, \tag{227}$$

$$= \pm\, i\boldsymbol{\epsilon}_\pm \,. \tag{228}$$

Substituting this form into the Newtonian equations of motion

$$m\frac{d^2\boldsymbol{x}(t, \boldsymbol{x}_0)}{dt} = q\left(\boldsymbol{E}_0(t, \boldsymbol{x}_0) + \frac{\boldsymbol{v}(t, x)}{c} \times \mathcal{B}_0\hat{\boldsymbol{z}}\right) , \tag{229}$$

we find

$$-\,m\omega^2\, x_\omega = qE_0 e^{ikz} \pm \omega\frac{q}{c}\mathcal{B}_0 x_\omega \,, \tag{230}$$

Minor manipulations yield

$$x_\omega = -\frac{qE_0 e^{ikz}}{m\omega}\frac{1}{\omega \pm \Omega_c} \,, \tag{231}$$

where $\Omega_c = q\mathcal{B}_0/mc$ is the cyclotron frequency. The induced current is

$$\boldsymbol{j}_\omega = n_0 q(-i\omega x_\omega)\boldsymbol{\epsilon}_\pm \,, \tag{232}$$

$$= \left[-\frac{\omega_p^2}{\omega(\omega \pm \Omega_c)}\right]\left(-i\omega E_0 e^{ikz}\boldsymbol{\epsilon}_\pm\right) \,. \tag{233}$$

(f) Following the logic of part (c) the required dispersion relation is

$$k_\pm(\omega) = \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{\omega(\omega \pm \Omega_c)}\right]^{1/2} \,. \tag{234}$$

For a $\omega \gg \omega_p \sim \Omega_c$ we have

$$k_\pm(\omega) = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{2\omega^2} \pm \frac{\omega_p^2\Omega_c}{2\omega^3} + \ldots\right) , \tag{235}$$

and thus to order k^{-2} inclusive the dispersion relation reads

$$\omega_\pm(\boldsymbol{k}) = ck\left(1 + \frac{\omega_p^2}{2(ck)^2} \mp \frac{\omega_p^2\Omega_c}{2(ck)^3} + \ldots\right) \,. \tag{236}$$

Because the eigen frequencies of right-handed and left handed waves are different by a small amount, after a period of time ΔT the two waves will accumulate a small phase difference $\sim \omega_p^2\Omega_c/(ck)^2\Delta T$. The linearly polarized light will appear to slowly precess in time as it traverses the medium.

4.3 Angular momentum in a wave packet

Consider a wave packet with a transverse profile $E_o(x,y)$ propagating in the z direction (see eq. (239) for a complete specification of $\boldsymbol{E}$ and $\boldsymbol{B}$). Although the precise form of $E_o(x,y)$ is not needed below, for definiteness you may assume that the wave packet has a Gaussian profile for

$$E_o(x,y) = \mathcal{A}e^{-\frac{x^2+y^2}{4\sigma^2}}\,, \tag{237}$$

and is infinitely broad in the z direction. The following integrals may be useful:

$$\int_{-\infty}^{\infty} \mathrm{d}u\; e^{-\alpha u^2} = \sqrt{\frac{\pi}{\alpha}}\,, \tag{238a}$$

$$\int_{-\infty}^{\infty} \mathrm{d}u\; e^{-\alpha u^2} e^{iku} = \sqrt{\pi} e^{-\frac{k^2}{4\alpha}}\,. \tag{238b}$$

(a) (2 points) When all derivatives of $E_o(x,y)$ are neglected, show that[6]

$$\boldsymbol{E}^{(0)}(t,\boldsymbol{r}) = E_o(x,y)\, e^{i(kz-\omega t)}\, \frac{(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})}{\sqrt{2}}\,, \tag{239a}$$

$$\boldsymbol{B}^{(0)}(t,\boldsymbol{r}) = \hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)}\,, \tag{239b}$$

is a solution to the Maxwell equations for $\omega = ck$.

(b) (3 points) Calculate the time averaged energy per length in the wave packet, $\langle U \rangle$.

(c) (5 points) When the derivatives of $E_o(x,y)$ are not neglected, Eq. (239) is not a solution to the Maxwell equations. Determine the corrections to $\boldsymbol{E}^{(0)}$ and $\boldsymbol{B}^{(0)}$ to first order in gradients for $k\sigma \gg 1$.

Hint: try a solution for $\boldsymbol{E}$ (and analogously for $\boldsymbol{B}$) of the form

$$\boldsymbol{E}(t,\boldsymbol{r}) = \boldsymbol{E}^{(0)} + E^{(1)}(x,y)e^{i(kz-\omega t)}\hat{\boldsymbol{z}}\,, \tag{240}$$

and determine the correction $E^{(1)}(x,y)$ in terms of $E_o(x,y)$ and its derivatives.

(d) (4 points) Write the solution to part (c) as a linear superposition of the plane wave solutions to the Maxwell equations. First use the superposition to qualitatively explain the correction to the electric field (proportional to $\hat{\boldsymbol{z}}$), and then use the superposition to precisely reproduce this correction.

(e) (4 points) Calculate the z-component of the time averaged angular momentum per length in the wave packet, $\langle L^z \rangle$, to the lowest non-trivial order in $k\sigma$.

(f) (2 points) Determine the ratio $\langle L^z \rangle / \langle U \rangle$. Interpret the result using photons.

[6] This is Gaussian or Heaviside-Lorentz units. In SI units the magnetic field reads, $\boldsymbol{B}^{(0)} = \frac{1}{Z_0}\hat{\boldsymbol{z}} \times \boldsymbol{E}^{(0)}$ where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376\,\text{Ohms}$ is the vacuum impedance.

Solution

(a) The Maxwell equations in free space read

$$\nabla \cdot \boldsymbol{E} = 0\,, \tag{241a}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = 0\,, \tag{241b}$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{241c}$$

$$-\frac{1}{c}\partial_t \boldsymbol{B} - \nabla \times \boldsymbol{E} = 0\,. \tag{241d}$$

Substituting

$$\boldsymbol{E} = E_o e^{i(kz-\omega t)} \boldsymbol{\epsilon}_+\,, \tag{242a}$$

$$\boldsymbol{B} = E_o e^{i(kz-\omega t)} \hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+\,, \tag{242b}$$

with E_o constant, gives the conditions

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\epsilon}_+ = 0\,, \tag{243a}$$

$$\frac{i\omega}{c}\boldsymbol{\epsilon}_+ + ik\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) = 0\,, \tag{243b}$$

$$\hat{\boldsymbol{z}} \cdot (\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) = 0\,, \tag{243c}$$

$$\frac{i\omega}{c}(\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) - ik(\hat{\boldsymbol{z}} \times \boldsymbol{\epsilon}_+) = 0\,. \tag{243d}$$

These equations are all clearly satisfied if $\omega = ck$ and $\boldsymbol{\epsilon}_+ = (\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})/\sqrt{2}$.

(b) The time-averaged energy per length is

$$U = \int dx \int dy\, \frac{1}{2}\left\langle \boldsymbol{E}^2 + \boldsymbol{B}^2 \right\rangle\,, \tag{244}$$

$$= \int dx \int dy\, \frac{1}{4}\left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2 \right) \tag{245}$$

$$= \frac{1}{2}\int dx \int dy\, (E_o(x,y))^2\,, \tag{246}$$

$$= \mathcal{A}^2(\pi\sigma^2)\,. \tag{247}$$

We used the fact that

$$|\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}|^2 = 1 \qquad |\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})|^2 = 1 \tag{248}$$

We also used the "time-averaging theorem", i.e. that the time average of two harmonically varying quantitites is

$$\langle A(t)B(t)\rangle \equiv \left\langle \mathrm{Re}[Ae^{-i\omega t}]\, \mathrm{Re}[Be^{-i\omega t}] \right\rangle = \frac{1}{2}\mathrm{Re}[AB^*] \tag{249}$$

which greatly simplifies all practical computations in EM.

(c) We need to satisfy

$$\nabla \cdot \boldsymbol{E} = 0 \,. \tag{250}$$

Substituting the suggested ansatz, this equation reads

$$\frac{1}{\sqrt{2}}\partial_x E_o(x,y) + \frac{i}{\sqrt{2}}\partial_y E_o(x,y) + ikE^{(1)}(x,y) = 0 \,, \tag{251}$$

and thus

$$E^{(1)} = \frac{i}{\sqrt{2}k}\left(\frac{\partial E_o}{\partial x} + i\frac{\partial E_o}{\partial y}\right) \,. \tag{252}$$

For the magnetic field, we have

$$\boldsymbol{B}^{(0)} = E_o \frac{(-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})}{\sqrt{2}} e^{i(kz-\omega t)} \,. \tag{253}$$

So since $\nabla \cdot \boldsymbol{B} = 0$,

$$\frac{-i}{\sqrt{2}}\partial_x E_o + \frac{1}{\sqrt{2}}\partial_y E_o + ikB^{(1)} = 0 \,, \tag{254}$$

we find

$$B^{(1)} = \frac{i}{\sqrt{2}k}\left(-i\frac{\partial E_o}{\partial x} + \frac{\partial E_o}{\partial y}\right) \,. \tag{255}$$

(d) A general superposition (which is a pure plane in the z-direction) can be written

$$\boldsymbol{E}(t,\boldsymbol{r}) = \sum_{s=\pm} \int \frac{dk_x dk_y}{(2\pi)^2} E_o(\boldsymbol{k}, s) e^{i(k_x x + k_y y + k_z z - \omega(\boldsymbol{k})t)} \boldsymbol{\epsilon}_s(\boldsymbol{k}) \,, \tag{256}$$

where

$$\boldsymbol{k} \cdot \boldsymbol{\epsilon}_s(\boldsymbol{k}) = 0 \,, \tag{257}$$

$$\boldsymbol{\epsilon}_s(\boldsymbol{k}) \cdot \boldsymbol{\epsilon}_s^*(\boldsymbol{k}) = 1 \,, \tag{258}$$

and of course

$$\omega(\boldsymbol{k}) = c\sqrt{k_x^2 + k_y^2 + k_z^2} \,. \tag{259}$$

It is always understood that the real part of Eq. (256) should be taken. The superposition we described above has

$$k_z \gg k_x, k_y \sim \frac{1}{\sigma} \,,$$

and is nearly circularly polarized. Qualitatively it is easy to see the need for a longitudinal correction to $\boldsymbol{E}^{(0)}$. The wave packet is a super-position of Fourier modes, one of which is shown in Fig. 4. The electric field points along the polarization vector, $\boldsymbol{\epsilon}(\boldsymbol{k})$. Sine the polarization vector is perpendicular to $\boldsymbol{k}$, it points partly in the z direction when k_x and k_y are non-zero. Thus, there must be a component of the electric field in the z-direction. We will now show how this reasoning quantitatively reproduces part (b).

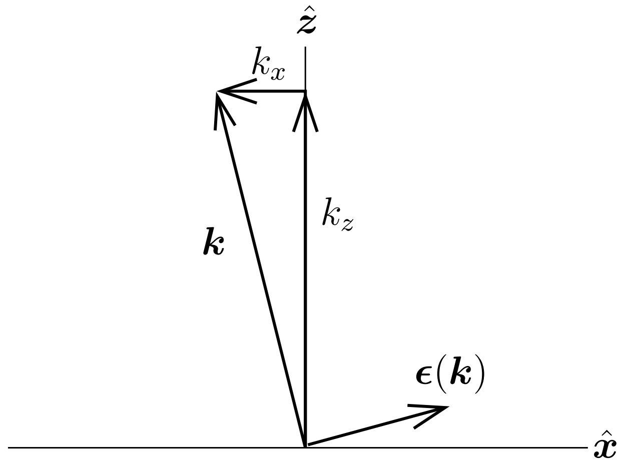

Figure 4: A typical Fourier mode in the wave packet and its polarization vector $\boldsymbol{\epsilon}(\boldsymbol{k})$.

First we note that to linear order in $k_\perp/k_z$

$$k = \sqrt{k_\perp^2 + k_z^2} \simeq k_z \, , \tag{260}$$

implying that $\omega = ck \simeq ck_z$ are all approximately constant, and may be brought out of the integral in Eq. (256). We next decompose $\boldsymbol{k}$ and $\boldsymbol{\epsilon}$ into components transverse and parallel to $\hat{\boldsymbol{z}}$

$$\boldsymbol{k} \equiv \vec{k}_\perp + k_z \hat{\boldsymbol{z}} \, , \tag{261}$$

$$\boldsymbol{\epsilon} \equiv \vec{\epsilon}_\perp + \epsilon_z \hat{\boldsymbol{z}} \, . \tag{262}$$

Intuition from the plane wave solutions says that $|\vec{\epsilon}_\perp| \gg \epsilon_z$. Indeed, from the orthogonality condition

$$\vec{k}_\perp \cdot \vec{\epsilon}_\perp + k_z \epsilon_z = 0 \tag{263}$$

we find that

$$-\frac{\vec{k}_\perp \cdot \vec{\epsilon}_\perp}{k} = \epsilon_z \, . \tag{264}$$

The distribution is therefore

$$\boldsymbol{E}(t, \boldsymbol{r}) = e^{i(kz - \omega t)} \int \frac{dk_x dk_y}{(2\pi)^2} E_o(k_x, k_y) e^{i(k_x x + k_y y)} \left(\vec{\epsilon}_\perp - \frac{\vec{k}_\perp \cdot \vec{\epsilon}_\perp}{k} \hat{\boldsymbol{z}} \right) \, . \tag{265}$$

Taking $\vec{\epsilon}_\perp = (\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})/\sqrt{2}$, and using the properties of Fourier transforms, i.e.

$$\underbrace{ik_j}_{\text{Fourier space}} \quad \leftrightarrow \quad \underbrace{\partial_j}_{\text{coordinate space}} \, , \tag{266}$$

yields

$$\boldsymbol{E}(t, \boldsymbol{r}) = e^{ikz - i\omega t} \left(E_o(x, y) \frac{(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})}{\sqrt{2}} + \frac{i}{\sqrt{2}k} \left(\partial_x E_o(x, y) + i \partial_y E_o(x, y) \right) \hat{\boldsymbol{z}} \right) \, . \tag{267}$$

Clearly we want

$$E_o(x,y) = \mathcal{A}e^{-(x^2+y^2)/(4\sigma^2)}\,, \tag{268}$$

and thus

$$E_o(k_x,k_y) = \int \mathrm{d}x\,\mathrm{d}y\, E_o(x,y)\, e^{-ik_x x - ik_y y} = \mathcal{A}\,(4\pi\sigma^2)\, e^{-\sigma^2(k_x^2+k_y^2)}\,, \tag{269}$$

fully specifying the fourier decomposition in Eq. (265).

(e) The time averaged angular momentum per length is

$$\boldsymbol{L} = \frac{1}{c}\int dx \int dy\, \langle \boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B})\rangle\ . \tag{270}$$

The integrand of the z-component of the angular momentum involves

$$\hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}) = (\hat{\boldsymbol{z}}\cdot\boldsymbol{E})(\boldsymbol{r}\cdot\boldsymbol{B}) - (\boldsymbol{r}\cdot\boldsymbol{E})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B})\,. \tag{271}$$

We see that because of the $\hat{\boldsymbol{z}}\cdot\boldsymbol{E}$ and $\hat{\boldsymbol{z}}\cdot\boldsymbol{B}$ terms the angular momentum necessarily involves the first correction, $\boldsymbol{E}^{(1)}$ and $\boldsymbol{B}^{(1)}$. The time averaged angular momentum involves

$$\langle\hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}))\rangle = \frac{1}{2}\mathrm{Re}[(\hat{\boldsymbol{z}}\cdot\boldsymbol{E})(\boldsymbol{r}\cdot\boldsymbol{B})^*] - \frac{1}{2}\mathrm{Re}[(\boldsymbol{r}\cdot\boldsymbol{E})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B})^*]\,. \tag{272}$$

Straightforward steps yield

$$\begin{aligned}
\mathrm{Re}[(\hat{\boldsymbol{z}}\cdot\boldsymbol{E})(\boldsymbol{r}\cdot\boldsymbol{B})^*] =& \mathrm{Re}[(\hat{\boldsymbol{z}}\cdot\boldsymbol{E}^{(1)})(\boldsymbol{r}\cdot\boldsymbol{B}^0)^*]\,, & (273)\\
=& \mathrm{Re}[\frac{1}{2ik}\left(\partial_x E_o + i\partial_y E_o\right)\left(-ixE_o + yE_o\right)^*]\,, & (274)\\
=& \frac{1}{2k}\left(xE_o\partial_x E_o + yE_o\partial_y E_o\right)\,, & (275)\\
=& \frac{1}{4k}\left(x\partial_x E_o^2 + y\partial_y E_o^2\right)\,. & (276)
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{Re}[(\boldsymbol{r}\cdot\boldsymbol{E})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B})^*] =& \mathrm{Re}[(\boldsymbol{r}\cdot\boldsymbol{E}^{(0)})(\hat{\boldsymbol{z}}\cdot\boldsymbol{B}^1)^*]\,, & (277)\\
=& \mathrm{Re}\left[(xE_o + iyE_o)\frac{1}{2k}\left(\partial_x E_o + i\partial_y E_o\right)^*\right]\,, & (278)\\
=& \frac{1}{2k}\left(xE_o\partial_x E_o + yE_o\partial_y E_o\right)\,, & (279)\\
=& \frac{1}{4k}\left(x\partial_x E_o^2 + y\partial_y E_o^2\right)\,. & (280)
\end{aligned}$$

Thus

$$\begin{aligned}
\langle L^z\rangle =& \frac{1}{c}\int dx\int dy\,\langle\hat{\boldsymbol{z}}\cdot(\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}))\rangle\,, & (281)\\
=& \int dx\int dy \frac{1}{4ck}\left(x\partial_x E_o^2 + y\partial_y E_o^2\right)\,. & (282)
\end{aligned}$$

Integrating by parts we find

$$\langle L^z \rangle = \frac{1}{2ck} \int dx \int dy \, E_o^2 \,, \tag{283}$$

$$= \frac{1}{ck} \mathcal{A}^2 (\pi\sigma^2) \,. \tag{284}$$

(f) For the required ratio we find

$$\frac{\langle L^z \rangle}{\langle U \rangle} = \frac{1}{\omega} \,. \tag{285}$$

This is consistent with our quantum expectation. Each photon of definite frequency ω and wave number $\boldsymbol{k} \simeq \frac{\omega}{c} \hat{\boldsymbol{z}}$ carries energy $E = \hbar\omega$ and spin angular momentum $\hbar$:

$$\frac{\langle L^z \rangle}{\langle U \rangle} = \frac{\hbar}{\hbar\omega} \,. \tag{286}$$