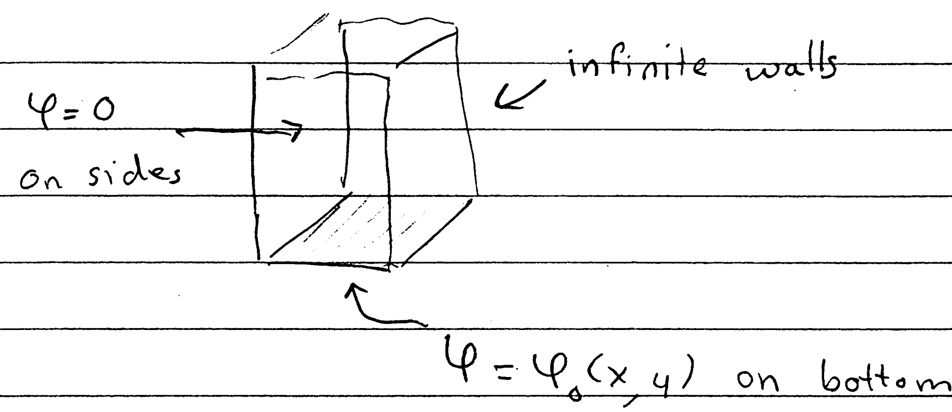
Separation of Variables

Cartesian Components (Jackson 2.9)

Consider the following boundary value problem



Then try a separated solution $\Phi = X(x)\,Y(y)\,Z(z)$

$$-\frac{1}{\varphi}\nabla^2 \varphi = 0 \qquad \text{in interior}$$

This leads to

$$\underbrace{-\frac{1}{X}\frac{d^2X}{dx^2}} + \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2} + -\frac{1}{Z}\frac{d^2Z}{dz^2}} = 0$$

if x changes, then these terms are constant

Conclude

$$-\frac{d^2X}{dx^2} = k_x^2\, X$$

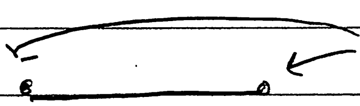
$$-\frac{d^2Y}{dy^2} = k_y^2\, Y$$

The // directions to the surface where the non-trivial (inhomogeneous) boundary condition $\varphi_0(x,y)$ is specified

$$-\frac{d^2Z}{dx^2} = -(k_x^2 + k_y^2)\, Z$$

the $\perp$ direction

So far k_x and k_y are arbitrary. But there are two boundary conditions specified on X



X is zero at $x=0$ and $x=a$

$x=0$ $x=a$

Further multiply X by a constant does not change the boundary value, i.e. zero of X.
Thus, for general k_x there is no solution.
For specific k_x the boundary conditions can be satisfied

The equation is thus an eigenvalue equation

$$-\frac{d^2X_n}{dx^2} = k_x^2\, X_n \quad \text{and} \quad -\frac{d^2Y_m}{dy^2} = k_y^2\, Y_m$$

The operator, $-\frac{d^2}{dx^2}$ with vanishing b.c., is self adjoint. The resulting eigenfunctions are complete and orthogonal. Thus, we have the following eigen functions

$$k_x = \frac{n\pi}{a} \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, \ldots$$

$$k_y = \frac{m\pi}{b} \quad \text{and} \quad Y_m(y) = \sin\left(\frac{m\pi y}{b}\right) \qquad m = 1, 2, \ldots$$

Then the solution in the perpendicular direction is

$$Z_{n,m} = A_{nm} e^{-\gamma_{nm} z} + B_{nm} e^{\gamma_{nm} z}$$

where $\gamma_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$

In summary, the solution takes the form $\psi_{nm} \equiv X_n \cdot Y_m$

$$\varphi(x,y,z) = \sum_{n,m} \underbrace{\left(A_{nm} e^{-\gamma_{nm} z} + B_{nm} e^{\gamma_{nm} z}\right)}_{\perp \text{ solution}} \underbrace{\psi_{nm}(x,y)}_{\parallel \text{ eigenfunction}}$$

The A's and the B's are adjusted to match the boundary conditions.

For the particular problem at hand:

$$\varphi \xrightarrow[r \to \infty]{} 0 \qquad \text{so} \quad B_{nm} = 0$$

Then

$$\varphi(x,y,z) = \sum_{nm} A_{nm}\, e^{-\gamma_{nm} z}\, \psi_{nm}(x,y)$$

Then at $z = 0$

$$\varphi\Big|_{z=0} = \varphi_0(x,y) = \sum A_{nm}\, \psi_{nm}(x,y)$$

Using the orthogonality of ψ_{nm}

$$A_{nm} = \left(\frac{2}{a}\right)\left(\frac{2}{b}\right) \int_0^a \int_0^b dx\,dy\; \psi^*_{nm}(x,y)\, \varphi_0(x,y)$$

Points to Take away

1. The $\parallel$ directions, x, y, are bounded on the sides and lead to eigenvalue equations
2. The $\perp$ direction, z, must be solved for each eigenmode
3. Using the completeness and orthogonality of the eigen-expansion, the coefficients can be adjusted to reproduce the desired boundary condition