Energy in Electrostatics. Jackson 1.11

$q_1 \cdot \quad \cdot q_2$

$q_3 \cdot \quad \cdot q_4$

The energy of a collection of charges $\{q_1, q_2\, q_3\, q_4\}$:

$$W_E = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{q_i q_j}{4\pi |\vec{x}_i - \vec{x}_j|}$$

The factor of $1/2$ is included becauses we sum over i and j rather than pairs of particles. W_E is the energy required to assemble the collection of charges.

In continuous form, the charge density is $\rho(\vec{x})$ and

$$\boxed{W_E = \frac{1}{2} \int_{\vec{x}} \int_{\vec{x}_0} \frac{\rho(\vec{x}\,)\,\rho(\vec{x}_0)}{4\pi |\vec{x} - \vec{x}_0|}} \qquad \int_{\vec{x}} \equiv \int d^3x$$

The potential is due to coulomb Law,

$$\varphi(x) = \int_{x_0} \frac{\rho(x_0)}{4\pi |\vec{x} - \vec{x}_0|}$$

So

$$\boxed{W_E = \frac{1}{2} \int_{\vec{x}} \rho(\vec{x})\, \varphi(\vec{x})}$$

Now use the Poisson equation

$$-\nabla^2 \varphi(\vec{x}) = \rho$$

Then

$$W_E = \frac{1}{2}\int_{\vec{x}} [-\partial_i \partial^i \varphi(x)]\, \varphi(x)$$

Integrating by parts (see below for further detail)

$$W_E = \frac{1}{2}\int_{\vec{x}} \underbrace{[-\partial^i \varphi(x)]}_{E^i(x)}\, \underbrace{[-\partial_i \varphi(x)]}_{E_i(x)}$$

and find

$$\boxed{W_E = \frac{1}{2}\int_x E^2(x)}$$

Thus we see that the electrostatic energy <u>density</u> is

$$\boxed{w_E = \frac{1}{2} E^2}$$

<u>Details</u>: we write $(-\partial_i \partial^i \varphi)\varphi = \underbrace{-\partial_i(\partial^i \varphi\varphi)}_{-\vec{\nabla}\cdot(\vec{\nabla}\varphi\varphi)} + \partial^i\varphi\partial_i\varphi$

a divergence

Then using the divergence theorem:

$$\int_V -\partial_i(\partial^i\varphi\varphi) + \partial^i\varphi\partial_i\varphi = \int_{\partial V} da_i \cdot (\partial^i\varphi\varphi) + \int_V \partial^i\varphi\partial_i\varphi$$

The surface integral $\longrightarrow 0$ as the volume becomes large, since the fields fall sufficiently rapidly as $r \to \infty$, i.e. $E \xrightarrow[r\to\infty]{} 0$

So:

$$\int_x \left[-\partial_i \partial^i \varphi(x) \right] \left[\varphi(x) \right] = \int_x \partial^i \varphi \, \partial_i \varphi$$

The general rule when integrating by parts (and throwing away surface terms) is to move the derivative from one term to the other and flip sign

$$(-\partial_i \partial^i \varphi)\varphi \longrightarrow (\partial^i \varphi)(\partial_i \varphi)$$