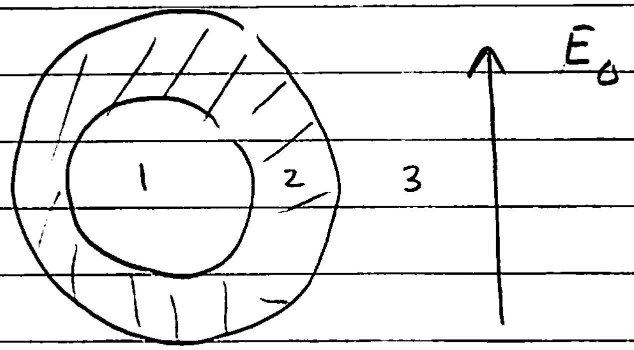
Problem Electric Field in a Spherical Shell



We have three regions. Each region satifies the laplace Eqn

$$\varphi_1 = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell$$

$$\varphi_2 = \sum_\ell \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right) P_\ell$$

$$\varphi_3 = \sum \left(F_\ell r^\ell + \frac{G_\ell}{r^{\ell+1}}\right) P_\ell$$

Boundary conditions give $B_\ell \longrightarrow 0$ for regularity.
Similarly $F_1 \xrightarrow[r \to \infty]{} -E_0$, & all other $F_\ell = 0$.

The last B.C. comes from the requirement that:

$$\varphi_3 \xrightarrow[r \to \infty]{} -E_0 r \cos\theta$$

Spherical Shell pg 2

So since only $\ell = 1$ is sourced by the boundary conditions we will look for a solution involving $\ell = 1$ and set all othe multipoles to zero

$$\varphi_1 = A r \cos\theta$$

$$\varphi_2 = C r \cos\theta + \frac{D}{r^2}\cos\theta$$

$$\varphi_3 = -E_0 r \cos\theta + \frac{G}{r^2}\cos\theta$$

The boundary conditions are

$$E_2^{\parallel} = E_1^{\parallel}\Big|_{r=a} \qquad E_3^{\parallel} = E_2^{\parallel}\Big|_{r=b}$$

$$D_2^{\perp} = D_1^{\perp}\Big|_{r=a} \qquad D_3^{\parallel} = D_2^{\parallel}\Big|_{r=b}$$

The $E^{\parallel}$ boundary conditions gives continuity of φ across $r=a$ and across $r=b$

$$\boxed{\begin{aligned} Aa &= Ca + \frac{D}{a^2} \\ Cb + \frac{D}{b^2} &= -E_0 b + \frac{G}{b^2} \end{aligned}}$$

Spherical Shell 3.

The remaining condition

$$-\varepsilon \left.\frac{\partial \varphi_2}{\partial r}\right|_{r=a} = -\left.\frac{\partial \varphi_1}{\partial r}\right|_{r=a}$$

$$-\left.\frac{\partial \varphi_3}{\partial r}\right|_{r=b} = -\varepsilon \cdot \left.\frac{\partial \varphi_2}{\partial r}\right|_{r=b}$$

Gives:

$$\boxed{\begin{aligned} &+\varepsilon\left(C - \frac{2D}{a^3}\right) = A \\ &-E_0 - \frac{2G}{b^3} = \varepsilon\left(C - \frac{2D}{b^3}\right) \end{aligned}}$$

These four equations are sufficient to determe A, C, D, G

Spherical Shell 4.

b) For $a \to 0$ for regularity we demand:

$$D \to 0$$

Then our eqs come from only the 2,3 interface:

$$(1) \quad Cb = -E_0 b + \frac{G}{b^2}$$

$$(2) \quad -E_0 - \frac{2G}{b^3} = \varepsilon\left(C - \cancelto{0}{\frac{2D}{b^3}}\right)$$

So solving these two eqs for C and G:

$$C = -E_0 + \frac{G}{b^3}$$

Then from (2)

$$-E_0 - \frac{2G}{b^3} = \varepsilon\left(-E_0 + \frac{G}{b^3}\right)$$

$$+(\varepsilon - 1)E_0 = (\varepsilon + 2)\frac{G}{b^3}$$

$$\left(\frac{\varepsilon - 1}{\varepsilon + 2}\right) b^3 E_0 = G$$

Then

$$C = -E_0 + \left(\frac{\varepsilon-1}{\varepsilon+2}\right)\frac{b^3}{b^3}E_0$$

So

$$C = \left(\frac{\varepsilon-1}{\varepsilon+2} - 1\right)E_0$$

$$\boxed{C = \frac{-3}{\varepsilon+2}E_0}$$

So the potential in the sphere

$$\varphi = C r\cos\theta$$

$$\varphi_2 = -\frac{3r\cos\theta}{\varepsilon+2}E_0 = \frac{-3}{\varepsilon+2} z E_0$$

So the electric field:

$$\vec{E} = -\frac{\partial\varphi}{\partial z}\hat{z}$$

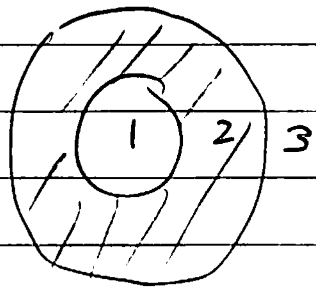
$$\boxed{\vec{E} = \frac{3E_0\hat{z}}{\varepsilon+2}}$$

for $\varepsilon \to 1$ $\vec{E} = E_0\hat{z}$,

$\varepsilon \to \infty$ $E = 0$

Not part of exam - Spherical Shell

To solve the system of four equations we recognize that if we know A_1, B_1



$$\varphi_1 = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

$$\varphi_2 = \left(A_2 r + \frac{B_2}{r^2}\right)\cos\theta$$

$$\varphi_3 = \left(A_3 r + \frac{B_3}{r^2}\right)\cos\theta$$

then we can find A_2, B_2

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} & M_{21} & \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$$

Similarly if we know A_2, B_2 then we can find $A_3\ B_3$

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} & M_{32} & \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$$

Not part of Exam - spherical Shell

The equations for matching $1 \to 2$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} M_{21} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

read

$$A = C + D/a^3$$

$$A = \varepsilon C - 2D/a^3 \varepsilon$$

$$\frac{(2\varepsilon+1)}{3\varepsilon} A = C$$

$$\frac{(\varepsilon - 1)}{3\varepsilon} A = D/a^3$$

And the equations matching $3 \to 2$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} M_{23} \end{pmatrix} \begin{pmatrix} -E_0 \\ G \end{pmatrix}$$

$$C + \frac{D}{b^3} = -E_0 + \frac{G}{b^3}$$

$$\varepsilon C - \frac{2\varepsilon D}{b^3} = -E_0 - \frac{2G}{b^3}$$

Not part of exam spherical shell

$$C = \left[-(2\varepsilon+1) E_0 + \frac{2(\varepsilon-1) G}{b^3} \right] \frac{1}{3\varepsilon}$$

$$\frac{D}{b^3} = \left[-(\varepsilon-1) E_0 + (\varepsilon+2) \frac{G}{b^3} \right] \frac{1}{3\varepsilon}$$

☆ $$(2\varepsilon+1) A = -(2\varepsilon+1) E_0 + 2(\varepsilon-1) \tilde{G}$$

☆☆ $$\frac{a^3}{b^3} (\varepsilon-1) A = \left[-(\varepsilon-1) E_0 + (\varepsilon+2) \tilde{G} \right]$$

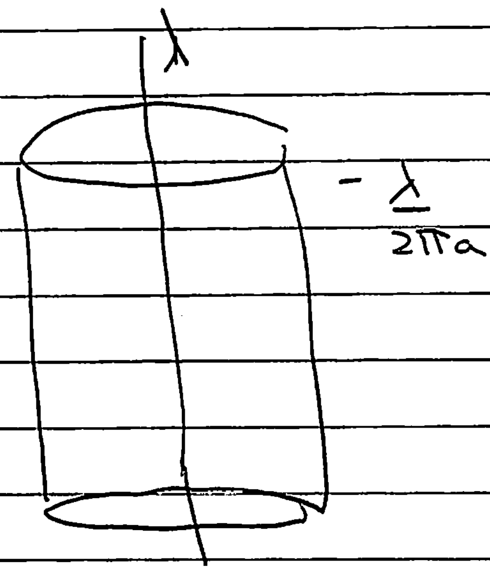
So multiplying ☆ by $(\varepsilon+2)$, and ☆☆ by $2(\varepsilon+1)$ we have

$$\left[(\varepsilon+2)(2\varepsilon+1) + \frac{2a^3}{b^3} (\varepsilon-1)^2 \right] A = \left[-(\varepsilon+2)(2\varepsilon+1) + 2(\varepsilon-1)^2 \right] E_0$$

$$\boxed{A = \frac{-9\varepsilon E_0}{\left[\cancel{(1+2\varepsilon)(2+\varepsilon)} - \frac{2a^3}{b^3} (\varepsilon-1)^2 \right]}}$$

$$\tilde{G} = \frac{G}{b^3} = \frac{E_0 (\varepsilon-1)(1+2\varepsilon)(1 - a^3/b^3)}{\left[\cancel{(1+2\varepsilon)(2+\varepsilon)} - \frac{2a^3}{b^3} (\varepsilon-1)^2 \right]}$$

Induced Rotation



From Gauss Law:

$$\int \vec{E} \cdot d\vec{a} = Q_{enc}$$

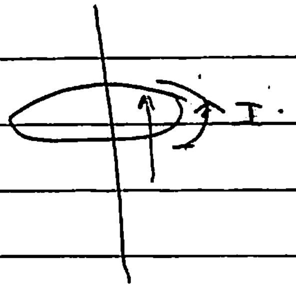
$$E_r \, 2\pi r_\perp L = \lambda L$$

$$E_r = \frac{\lambda}{2\pi r_\perp} \qquad r_\perp < a$$

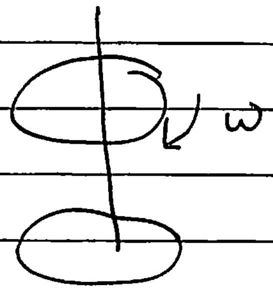
So outside gauss law gives $E = 0$

Induced Rotation pg. 2

b) The changing magnetic field induces currents which tends to support the decreasing field. The RH rule shows we want the currents to go counter clockwise (when looked at from above)



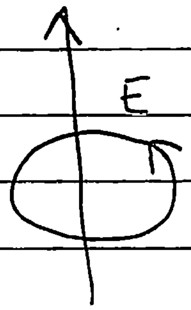
Since the cylinder is negatively changed we want the cylinder to rotate in a clockwise fashion



c) From Faraday Law

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

we have



$$\int \vec{E}\cdot d\vec{l} = -\frac{\partial}{c\,\partial t}\int \vec{B}_0 \cdot z \; da$$

$$E_\phi \cdot 2\pi r_\perp = -\frac{\partial}{c\,\partial t} B_0 \left(1 - \frac{t}{T}\right) \pi r_\perp^2$$

$$E_\phi = \frac{B}{c}\,\frac{r_\perp}{2T}$$

So the electric field is

$$E_\phi \Big|_{r=a} = \frac{B}{c}\,\frac{a}{2T}$$

So the force on the cylinder (per length)

$$F_\phi = E_\phi \cdot \cancel{2\pi a}\left(\frac{-\lambda}{\cancel{2\pi a}}\right)$$

$$F_\phi = E_\phi(-\lambda)$$

So the torque is

$$rF_\phi = aE_\phi(-\lambda) = I\alpha$$

and $$\alpha = \frac{B}{c\,2T}\,\frac{a^2(-\lambda)}{I}$$

Then

$$\omega = \int_0^t \alpha\, t$$

$$\omega = \begin{cases} 0 & t < 0 \\ \dfrac{B a^2 (-\lambda)}{2cTI}\, t & 0 < t < T \\ \dfrac{Ba^2 (-\lambda)}{2cI} & t > T \end{cases}$$

d) Then when we solved for the induced electric field

$$\nabla \times E = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

we treated $\vec{B}$ as $\vec{B}_0$ neglecting the fact that $\vec{E}$ makes $\vec{B}$.

$$\nabla \times B^{(ind)} = +\frac{1}{c}\frac{\partial E}{\partial t}$$

This uses the fact that B^{ind} is smaller than B_0 by $\left(\frac{a}{cT}\right)^2$

Induced Rotation pg. 5

indeed using that we found

$$E \sim \frac{a}{cT} B_0$$

Then

$$\nabla \times B^{ind} \sim \frac{1}{c} \frac{\partial E}{\partial t}$$

$$\frac{1}{a} B^{ind} \sim \frac{1}{cT} \frac{a B_0}{cT}$$

$$B^{ind} \sim \left(\frac{a}{cT}\right)^2 B_0$$

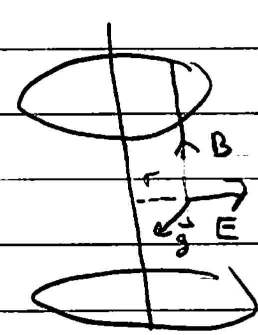
e) Then the angular momentum for $t > T$ is (this is the angular momentum per length)

$$\frac{\vec{L}}{h} = I \vec{\omega} = -\frac{B a^2}{2c} \lambda \, \hat{z}$$

height of cylinder

In the initial field configuration

$$\vec{L} = \int d^3r \; \vec{r} \times \vec{g}$$



$$\hat{r} \times \hat{g} = -\hat{z}$$

$$\vec{L} = h \int d^2r \; r_\perp \frac{E(r) B_0}{c} (-\hat{z})$$

So

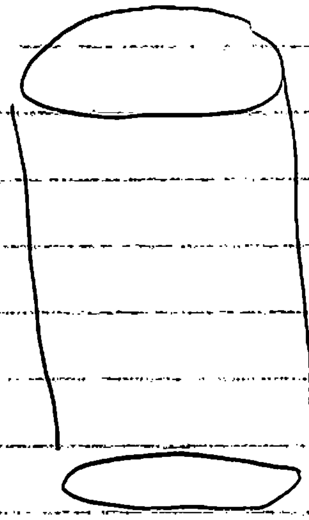
$$\frac{\vec{L}}{\hbar} = -\hat{z} \int \cancel{2\pi} r_\perp \, dr_\perp \, \cancel{r_\perp} \frac{\lambda}{\cancel{2\pi} \cancel{r_\perp}} \frac{B_0}{c}$$

$$\frac{\vec{L}}{\hbar} = -\hat{z}\, B_0 \frac{\lambda}{2c} r_\perp^2 \Big|_0^a$$

$$\boxed{\frac{\vec{L}}{\hbar} = -\hat{z}\, B_0 \frac{\lambda}{2c} a^2}$$

which agrees w/ before

Currents in a cylindrical Shell



$$\vec{K} = K_0 \cos 2\phi \ \hat{z}$$

We will solve this with the vector potential

$$-\nabla^2 A_z(\rho,\phi) = 0 \quad \text{outside and inside}$$

we will use the B.C

$$n \times (B_2 - B_1) = \vec{K}/c$$

$$n \cdot (B_2 - B_1) = 0$$

To match the solutions across the jump

Currents in a shell pg. 2

From

$$-\nabla^2 = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_z}{\partial\rho} + -\frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2}$$

We solve inside and out (ω)

(in) $$A_z = C_0 \ln\rho + D_0 + \sum_l \left(C_l \rho^l + \frac{D_l}{\rho^l}\right)\cos l\phi$$

(out) $$A_z = E_0 \ln\rho + F_0 + \sum_l \left(E_l \rho^l + \frac{F_l}{\rho^l}\right)\cos l\phi$$

Since the Boundary conditions only source the $l=2$ harmonics (i.e. $\vec{K} = K_0 \cos 2\phi$) we will try a solution

$$A_z^{in} = \left(C\rho^2 + \cancelto{0}{\frac{D}{\rho^2}}\right)\cos 2\phi$$

$$A_z^{out} = \left(\cancelto{0}{E\rho^2} + \frac{F}{\rho^2}\right)\cos 2\phi$$

we have set E and D to zero demanding regularity at $\rho \to 0$ and $\rho \to \infty$

Currents in a shell pg. 3

Now we use the boundary conditions to fix C and F

$$(\nabla \times A)^{in} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\hat{\rho} - \frac{\partial A_z}{\partial \rho}\hat{\phi}\right]$$

So

$$B^{in} = -2C\rho \sin 2\phi \; \hat{\rho} + -2C\rho \cos 2\phi \; \hat{\phi}$$

$$B^{out} = -2\frac{F}{\rho^3}\sin 2\phi \; \hat{\rho} + +2\frac{F}{\rho^3}\cos 2\phi \; \hat{\phi}$$

From the B.C.

$$\vec{n}\cdot(B_2 - B_1)\Big|_{\rho=a} = 0$$

$$-2Ca = -2\frac{F}{a^3}$$

And from B.C

$$\vec{n}\times(\vec{B}_2 - \vec{B}_1) = \frac{K_0}{c}\cos 2\phi \; \hat{z}$$

So

$$\frac{2F}{a^3} - \left(-2Ca\right) = K_0/c$$

$$\frac{2F}{a^3} - \left(\frac{-2F}{a^3}\right) = K_0/c$$

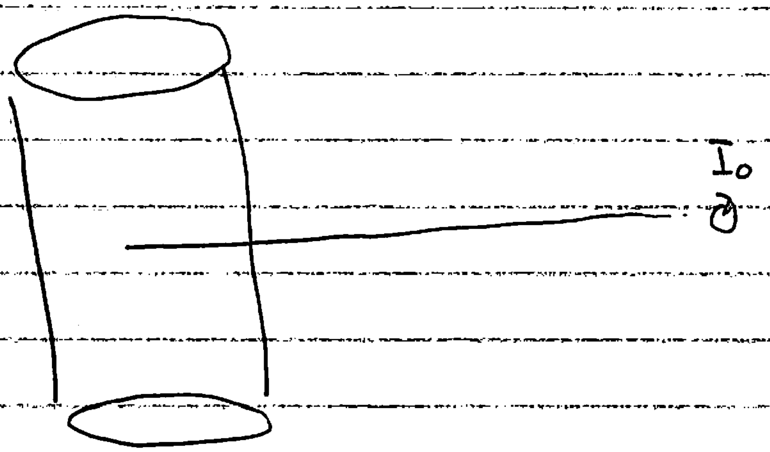
$$\frac{4F}{a^3} = K_0 \Rightarrow F = a^3\frac{K_0}{4}$$

So $\quad C = \dfrac{K_0}{4a}$

Then

$$\boxed{\begin{aligned} B^{in} &= -\frac{K}{2ac}\rho\sin 2\phi\,\hat{\rho} - \frac{K_0}{2ac}\rho\cos 2\phi\,\hat{\phi} \\ B^{out} &= -\frac{K_0}{2c}\left(\frac{a}{\rho}\right)^3\sin 2\phi\,\hat{\rho} + \frac{K_0}{2c}\left(\frac{a}{\rho}\right)^3\cos 2\phi\,\hat{\phi}\end{aligned}}$$

b) Then the force is



The magnetic moment is $\frac{I_0}{c} \pi r_0^2 \, (-\hat{x}) = \vec{m}$

The force on the current loop is

$$\vec{F} = (\vec{m} \cdot \nabla)\vec{B}$$

$$\vec{F} = -|\vec{m}| \frac{\partial}{\partial x} \vec{B}(\rho, \phi) \Big|_{\phi=0} \qquad \text{putting } \phi = 0$$

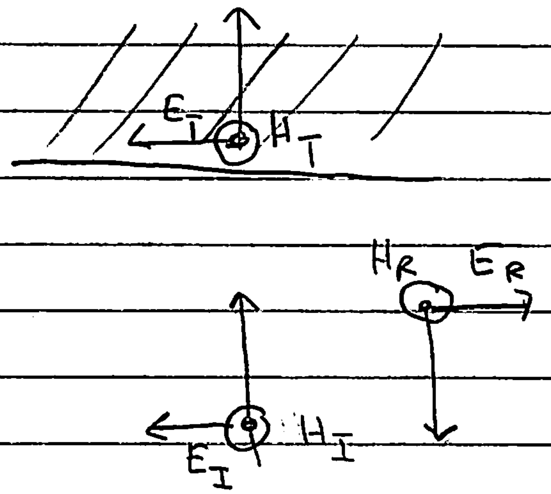
$$\vec{F} = -|\vec{m}| \frac{\partial}{\partial \rho} \vec{B}^{out} \Big|_{\phi=0} \qquad \hat{\phi} = \hat{y} \text{ at } \hat{\phi} = 0$$

$$= -\frac{I_0}{c} \pi r_0^2 \left(-\frac{3}{2} \frac{K_0}{c} \frac{a^3}{\rho^4} \cos 2\phi \right) \hat{y} \Big|_{\phi=0}$$

$$\boxed{\vec{F} = \frac{3}{2} \pi \frac{a^3}{\rho^4} r_0^2 \frac{K_0}{c^2} I_0 \hat{y}}$$

Reflection From Glass

Then from



$$\vec{E}_1 = -E_I\, e^{-i\omega t + ikz}\,\hat{x} + E_R\, e^{-i\omega t - ikz}\,\hat{x}$$

$$\vec{E}_2 = -E_T\, e^{-i\omega t + ikz}\,\hat{x}$$

So we have Boundary conditions

$$\vec{n} \times (\vec{E}_2 - \vec{E}_1)\Big|_{z=0} = 0$$

$$n \times (\vec{H}_2 - \vec{H}_1)\Big|_{z=0} = 0$$

The other B.C.

$$\left.\begin{aligned} \vec{n}\cdot(\vec{D}_2 - \vec{D}_1) &= 0 \\ n\cdot(\vec{B}_2 - \vec{B}_1) &= 0 \end{aligned}\right\} \text{These are trivially satisfied w/ } \vec{E} + \vec{B} \perp \text{ to } n$$

Reflection from glass pg: 2

So

$$-E_T = -E_I + E_R$$

The $\vec{H}$ conditions

$$i\vec{k} \times \vec{E} = \frac{i\omega}{c}\vec{B}$$

$$\frac{\hat{k}}{Z} \times \vec{E} = \vec{H} \qquad Z = \sqrt{\frac{1}{\varepsilon}}$$

Then the H conditions read

$$\vec{H}_1 = \left(E_I e^{ikz - i\omega t} + E_R e^{ikz - i\omega t}\right)\hat{y}$$

$$\vec{H}_2 = \frac{E_T}{Z} e^{ikz - i\omega t}\,\hat{y}$$

Yielding the B.C.

$$E_I + E_R = \frac{1}{Z} E_T$$

Reflection from glass pg. 3

Solving these two we have

$$-Z(E_I + E_R) = -E_I + E_R$$

$$(1 - Z)E_I = (1 + Z)E_R$$

$$\boxed{\frac{(1-Z)}{(1+Z)} E_I = E_R}$$

$$\boxed{\frac{(n-1)}{(n+1)} E_I = E_R}$$

and then E_T

$$E_T = Z(E_I + E_R)$$

$$= Z\left(1 + \frac{(1-Z)}{1+Z}\right)E_I$$

$$\boxed{E_T = \frac{2Z}{1+Z} E_I}$$

$$\boxed{E_T = \left(\frac{2}{n+1}\right) E_I}$$

Reflection from glass pg. 4

The transmission and reflection coefficents

$\vec{S} = c\,\vec{E} \times \vec{H}$ ← poynting Vector

$\vec{S} = \frac{c}{2}\frac{1}{z}|E|^2$ therefore $S_1 \propto |E|^2$ $S_2 \propto \frac{1}{z}|E|^2$

$\propto n|E|^2$

So since $1/z =$

$$T = \frac{\vec{S}_T \cdot \hat{z}}{S_I \cdot \hat{z}} = n\left(\frac{2}{n+1}\right)^2$$

While R

$$R = \frac{\vec{S}_R \cdot \hat{z}}{\vec{S}_I \cdot \hat{z}} = \left(\frac{n-1}{n+1}\right)^2$$

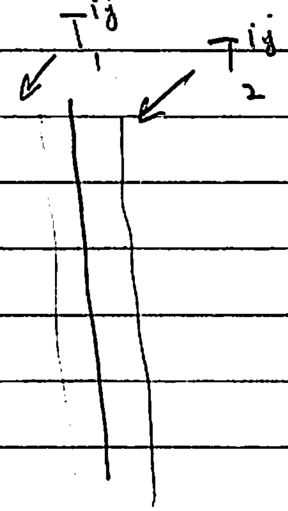
So

$$R + T = \frac{(n-1)^2}{(n+1)^2} + \left(\frac{2}{n+1}\right)^2 \cdot n = \frac{1}{(n+1)^2}\left((n-1)^2 + 4n\right)$$

$$= 1$$ ✓

Reflection from glass pg.5

c) Now lets evaluate the stress



The stress tensor

$$T_1^{ij} = -\varepsilon E^i E^j + \frac{1}{2}\delta^{ij} E^2$$
$$+ \; - B^i B^j + \frac{1}{2}\delta^{ij} B^2$$

$$\left\{ \begin{aligned} T_1^{xx} &= -\varepsilon E^x E^x + \frac{\delta^{xx}}{2} E^2 + \frac{\delta^{xx}}{2} B^2 \\ T_1^{yy} &= \frac{1}{2}\delta^{yy} E^2 - B^y B^y + \frac{\delta^{yy}}{2} |B|^2 \\ T_1^{zz} &= \frac{1}{2}\delta^{zz} E^2 + \frac{1}{2}\delta^{zz} B^2 . \end{aligned} \right.$$

Reflection from glass pg. 6

So for the reflected wave:

$$E_1^x = E_I (1-r) \qquad r \equiv \frac{n-1}{n+1}$$

$$H_1^y = E_I (1+r)$$

↳ amplitude reflection coefficient

So

$$\langle T^{xx} \rangle = -\frac{1}{2} E_I^2 (1-r)^2 + \frac{1}{4} E_I^2 (1-r)^2 + \frac{1}{4} E_I^2 (1+r)^2$$

$$= E_I^2 \left[-\frac{1}{2}(1-r)^2 + \frac{1}{4}(1-r)^2 + \frac{1}{4}(1+r)^2 \right]$$

$$= E_I^2 \left[-\frac{1}{2}(-2r) + \frac{1}{4}(-2r) + \frac{1}{4}(2r) \right]$$

$$\boxed{\langle T_1^{xx} \rangle = E_I^2 \, r}$$

$$\langle T^{yy} \rangle = \left[\frac{1}{4} E_I (1-r)^2 - \frac{1}{2} E_I (1+r)^2 + \frac{1}{4} E_I^2 (1+r)^2 \right]$$

$$\boxed{\langle T_1^{yy} \rangle = -E_I^2 \, r}$$

Reflection From glass pg. 7

$$T^{zz} = \delta^{zz} \frac{1}{4} E_I^2 (1-r)^2 + \frac{1}{4} \delta^{zz} E_I^2 (1+r)^2$$

$$\boxed{T_1^{zz} = \frac{1}{2} E_I^2 (1+r^2)}$$

amplitude transmission coefficient

$$t = \frac{2}{n+1}$$

From the transmitted stress:

$$T_2^{xx} = -\varepsilon E^x E^x + \frac{1}{2} \delta^{xx} \varepsilon E^2 + \frac{\delta^{yy}}{2} B^2$$

$$T_2^{yy} = +\frac{\delta^{yy}}{2} \varepsilon E^2 + -B^y B^y + \frac{\delta^{yy}}{2} B^2$$

$$T_2^{zz} = \frac{1}{2} \delta^{zz} E^2 + \frac{1}{2} \delta^{zz} B^2$$

$$H = B = \frac{1}{Z} E = \sqrt{\varepsilon} E$$

$$T_2^{xx} = -\frac{1}{2} \varepsilon E_I^2 t^2 + \frac{1}{4} \delta^{xx} \varepsilon E_I^2 t^2 + \frac{\delta^{yy}}{4} \varepsilon E_I^2 t^2$$

$$\boxed{T_2^{xx} = 0}$$

$$T^{yy} = \frac{1}{4} \delta^{yy} \varepsilon E_I^2 t^2 - \frac{1}{2} \varepsilon E_I^2 t^2 + \frac{1}{4} \delta^{yy} \varepsilon E_I^2 t^2$$

$$\boxed{T_2 = 0}$$

Reflection from glass 8

So

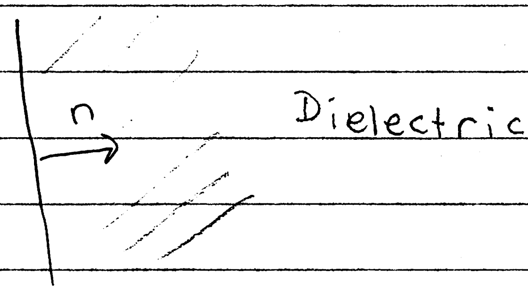
$$T_2^{zz} = \frac{1}{2}\delta^{zz}\varepsilon E^2 + \frac{1}{2}\delta^{zz} B^2$$

$$T_2^{zz} = \frac{1}{4}E_I^2\,\varepsilon t^2 + \frac{1}{4}E_I^2\,\varepsilon t^2$$

$$\boxed{T_2^{zz} = \frac{1}{2}E_I^2\, nT} \qquad \sqrt{\varepsilon}\, t^2 \equiv T$$

$$\sqrt{\varepsilon} \equiv n$$

So the The difference in the stress tensor tells us about the force



$-n_a \cdot (T_2^{ab} - T_1^{ab})$ = Force in the b-th direction

$$F^z_{net} = +T_1^{zz} - T_2^{zz}$$

$$F^z = \frac{1}{2}E_I^2\left[+(1+R) - nT\right] = -\frac{1}{2}E_I^2\left(\frac{n-1}{n+1}\right)$$

= Away from dielectric!