A Dielectric Sphere in an external field:

First note:

$$\varphi_{ext} = -E_0 r\cos\theta - \frac{1}{2}E_0'\Big(r^2\cos^2\theta - \frac{1}{2}r^2\underbrace{\sin^2\theta}_{1-\cos^2\theta}\Big)$$

$$= -E_0 r\cos\theta - \frac{1}{2}E_0' r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$= -E_0 r P_1(\cos\theta) - \frac{1}{2}E_0' r^2 P_2(\cos\theta)$$

Then we write:

$$\varphi_{in} = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$\varphi_{out} = \sum_l \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Then boundary conditions give, $B_l = 0$

and $C_1 = -E_0$

$C_2 = -\frac{1}{2}E_0'$ all other $C_l = 0$

(2) Sphere

Trying a solution with $l=1$ and $l=2$:

$$\varphi_{in} = A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta)$$

$$\varphi_{out} = C_1 r P_1(\cos\theta) + \frac{D_1}{r^2} P_1(\cos\theta)$$

$$+ C_2 r^2 P_2(\cos\theta) + \frac{D_2}{r^3} P_2(\cos\theta)$$

So then Boundary conditions $E_{\parallel}^{in} = E_{\parallel}^{out}$

$$E_{\parallel}^{in} = \frac{-1}{r}\frac{\partial \varphi_{in}}{\partial\theta} = \frac{-1}{r}\frac{\partial \varphi_{out}}{\partial\theta} = E_{\parallel}^{out}$$

Then this gives that:

(1) $$A_1 a = C_1 a + \frac{D_1}{a^2}$$

(2) $$A_2 a^2 = C_2 a^2 + \frac{D_2}{a^3}$$

Similarly then the boundary conditions on D

$$D_{\perp}^{in} = D_{\perp}^{out}$$

$$-\varepsilon \frac{\partial \varphi_{in}}{\partial r} = -\frac{\partial \varphi_{out}}{\partial r}$$

③ Sphere

So then in the P_1 components:

(3) $$-\varepsilon A_1 = -C_1 - (-2)\frac{D_1}{a^3}$$

and in the P_2 components:

(4) $$-2A_2 a\varepsilon = -2C_2 a - (-3)\frac{D_2}{a^4}$$

So counting equations and unknowns we have

Eqs (1), (2), (3), (4) for $A_1 D_1$ and $A_2 D_2$

Looking at the set for $A_1 D_1$ we have then

$$A_1 a = C_1 a + \frac{D_1}{a^2}$$

$$-\varepsilon A_1 = -C_1 + \frac{2D_1}{a^3}$$

Temporarily setting $a=1$, we solve for D_1:

$$\varepsilon A_1 = C_1\varepsilon + \varepsilon D$$

$$-\varepsilon A_1 = -C_1 + 2D$$

Then

$$0 = (\varepsilon-1)C_1 + (\varepsilon+2)D$$

$$\boxed{D_1 = -(\varepsilon-1)/(\varepsilon+2)\ C_1}$$

④ Sphere

Thus, similarly for the $l=2$ case we have then:

$$A_2 = C_2 + D_2$$

and so, $$-2A_2\varepsilon = -2C_2 + 3D_2$$

$$2\varepsilon A_2 = 2\varepsilon C_2 + 2\varepsilon D_2$$

$$0 = 2(\varepsilon-1)C_2 + (3+2\varepsilon)D_2$$

$$\boxed{-\frac{2(\varepsilon-1)}{3+2\varepsilon}C_1 = D_2}$$

Then solving for A_1

$$A_1 = C_1 + D_1$$

$$A_1 = \left(-\frac{(\varepsilon-1)}{\varepsilon+2} + 1\right)C_1$$

$$\boxed{A_1 = \frac{3}{\varepsilon+2}C_1}$$

and A_2 we have

$$A_2 = C_2 + D_2$$

$$= C_2\left(1 - \frac{2(\varepsilon-1)}{3+2\varepsilon}\right) = \boxed{\frac{5}{3+2\varepsilon}C_2 = A_2}$$

) Sphere

So

$$\boxed{\varphi_{in} = -\frac{3}{\varepsilon+2} E_0 r\cos\theta - \left(\frac{5}{3+2\varepsilon}\right)\frac{1}{2} E_0' r^2 P_2(\cos\theta)}$$

Then

$$\boxed{\begin{aligned}\varphi_{out} &= -E_0 r\cos\theta - \frac{1}{2}E_0' r^2 P_2(\cos\theta) \\ &+ \frac{(\varepsilon-1)}{(\varepsilon+2)} E_0 \frac{a^3}{r^2} P_1(\cos\theta) \\ &+ \frac{(\varepsilon-1)}{(3+2\varepsilon)} E_0' \frac{a^4}{r^3} P_2(\cos\theta)\end{aligned}}$$

Then we find for part b) :

$$\sigma = -(\cancel{P_2}^{\,0} - P_1)\cdot\vec{n}$$

$$= +P_1\cdot n \qquad \vec{P}_1 = (\varepsilon-1)\vec{E}_{in}$$

$$= +(\varepsilon-1)\left(-\frac{\partial\varphi_{in}}{\partial r}\right)$$

$$\boxed{\sigma = \frac{3(\varepsilon-1)}{\varepsilon+2} E_0 \cos\theta + \frac{3}{3+2\varepsilon}(\varepsilon-1) E_0' a \, P_2(\cos\theta)}$$

① Solenoid

Problem - Forces on a filled solenoid

a) $H_0(t) = H_0 e^{-i\omega t}$ $\quad H_{out} = 0$ $\quad \leftarrow$ notation $H_0(t) = H_0 e^{-i\omega t}$

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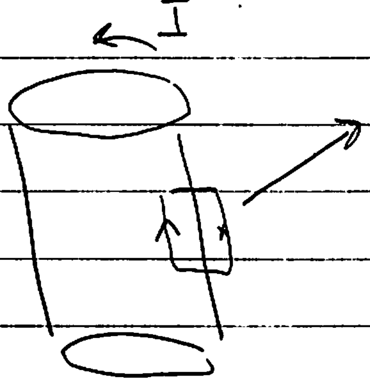


$$H_0(t) L = \frac{n I_0(t) L}{c}$$

$$\boxed{H_0(t) = \frac{n I_0(t)}{c}} \Rightarrow H_0 = \frac{n I_0}{c}$$

$$\boxed{B_0 = \mu H_0}$$

b) Then $\quad H = B - M \qquad M = (\mu - 1) H$

$$+\, n \times (\cancel{\vec{M}_2} - \vec{M}_1) = \frac{K_{mat}}{c}$$

$$-\, n \times \vec{M}_1 = \frac{\vec{K}_{mat}}{c}$$

on sides

$$\boxed{\hat{\phi}\, (\mu - 1) H_0(t) = \frac{\vec{K}_{mat}(t)}{c}}$$

c) Then computing the force per area:

$$T^{ab} = -E^a E^b + \delta^{ab} E^2/2 + \frac{1}{\mu}\left(-B^a B^b + \delta^{ab} B^2/2\right)$$

We care about $T^{\rho\rho}$

② Solenoid

Since $T^{\rho\rho}$ is only non-vanishing component of the stress in the $\hat{\rho}$ direction (the normal to the surface):

$$\frac{\text{Force}}{\text{Area}} = -\left(T^{\rho\rho}_{out} - T^{\rho\rho}_{in}\right)$$

$$= T^{\rho\rho}_{in} = \mu\left(-H^{\rho}H^{\rho} + \frac{\delta^{\rho\rho}}{2}H^2\right)$$

$$\frac{\text{Force}}{\text{Area}} = \frac{1}{2}\mu H_0^2(t)$$

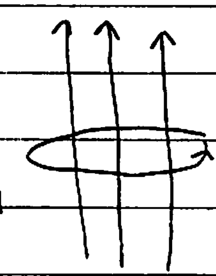
Time average $H_0(t) = H_0 e^{-i\omega t}$ gives a factor $1/2$

$$\boxed{\left\langle \frac{\text{Force}}{\text{Area}} \right\rangle = \frac{1}{4}\mu |H_0|^2}$$

d) Then to determine the electric field

$$-\nabla \times E = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

So in integral form:



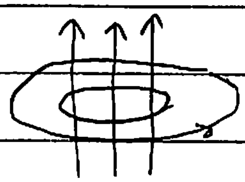
$$-\oint \vec{E}\cdot d\vec{l} = \frac{1}{c}\partial_t \Phi_B$$

$$-2\pi\rho E_\phi(t) = \frac{1}{c}\left(-i\omega B_0(t)\pi\rho^2\right)$$

$$E_\phi = \frac{+i\omega}{2c}\mu H_0(t)\rho \qquad \text{for } \rho < a$$

③ Solenoid

Outside we have



$$-\oint E\cdot dl = \frac{1}{c}\partial_t \Phi_B$$

$$-E_\phi 2\pi\rho = \frac{1}{c}\left(-i\omega\mu H_o(t)\,\pi a^2\right)$$

$$E_\phi = +\frac{i\omega}{2c}\frac{\mu H_o(t)\,a^2}{\rho} \qquad \text{outside}$$

- E is // to the surface and thus is continuous $E_{//}^{out} - E_{//}^{in} = 0$

e) For the quasistatic approximation we need $E^{ind} \ll B$. Comparing E_ϕ to $\mu H_o(t)$, we have for $\rho \sim a$

$$E_\phi \sim \left(\frac{\omega a}{c}\right)\mu H_o(t)$$

Thus, the quasistatic approximation is valid provided $\omega a/c \ll 1$

f) To find the correction to $H_o(t)$ we use the second maxwell equiation

$$\nabla \times \delta\vec{H}^{(2)} = \frac{1}{c}\partial_t \vec{E}$$

④ Solenoid

Using the fact that for $H_z(\rho)$:

$$(\nabla \times H)_\phi = -\frac{\partial H_z(t)}{\partial \rho} = -\frac{i\omega}{c} E_\phi(t)$$

So

$$-\frac{\partial H_z^{(2)}(t,\rho)}{\partial \rho} = -\frac{i\omega}{c} E_\phi(t,\rho)$$

$$-\cancelto{0}{H_z^{(2)}}\Big|_{\rho max} + H_z^{(2)}(\rho) = \int_\rho^{\rho max} d\rho \left(-\frac{i\omega}{c}\right)\left(\frac{+i\omega}{2c}\frac{a^2}{\rho}\right)\mu H_0(t)$$

$$H_z^{(2)}(\rho) = \frac{\omega^2 a^2}{2c^2}\mu H_0(t) \ln \frac{\rho_{max}}{\rho}$$

Setting $\rho = a$ we have

$$\boxed{H_z^{(2)}(a,t) = \frac{\omega^2 a^2}{2c^2}\mu H_0(t) \ln \rho_{max}/a}$$

Then, the continuity follows

$$n \times (H_{out}^{(0)} + H_{out}^{(2)} - (H_{in}^{(0)} + H_{in}^{(2)})) = \vec{K}_{mat}/c$$

But the zeroth order fields already satisfy the B.C.

$$n \times (H_{out}^{(0)} - H_{in}^{(0)}) = \vec{K}_{mat}/c$$

So we need

⑤ Solenoid

$$n \times \left(H^{(2)}_{out} - H^{(2)}_{in} \right) = 0$$

i.e. that $H^{(2)}$ is continuous across the interface

g) To find the correction to the force we need to compute the discontinuity in the electromagnetic stress.

$$\frac{F}{A} = -\left(T^{\rho\rho}_{out} - T^{\rho\rho}_{in}\right)$$

The electric field is continuous so it does not lead to a correction to the force.

$$T^{\rho\rho}_{H} = -H^{\rho}H^{\rho} + \frac{\delta^{\rho\rho}}{2} H^2$$

$$T^{\rho\rho}_{H} = \frac{\delta^{\rho\rho}}{2} H^2$$

Using that $H(t) = H_0(t) + \delta H$

$$\delta T^{\rho\rho} = \frac{1}{\cancel{2}} \left(\cancel{2} H_0(t)\, \delta H(t) \right) \leftarrow \text{so } \delta T^{\rho\rho} \text{ vanishes outside}$$

$H_0(t) = 0$ outside

So the discontinuity in the stress is

$$\frac{\delta F}{A} = -\left(\cancel{T^{\rho\rho}_{out}} - T^{\rho\rho}_{in} \right) = + \vec{H}_0(t) \cdot \delta \vec{H}(t)$$

⑥ Solenoid

Inserting the result from part (f) we have

$$\frac{\delta F}{A} = \frac{\omega^2 a^2}{c^2}\left(\frac{1}{2}\mu H_0^2(t)\right)\ln \rho_{max}/a$$

And the time average force is

$$\boxed{\left\langle \frac{F}{A} \right\rangle = \frac{1}{4}\mu H_0^2\left(1 + \frac{\omega^2 a^2}{c^2}\ln \rho_{max}/a + \ldots\right)}$$

①

Transmission

a)

$$\vec{E}(t,\vec{x}) = \vec{\mathcal{E}}\, e^{i\vec{k}\vec{x} - i\omega t}$$

(1) $$-\nabla \times E = \frac{1}{c}\partial_t \vec{B}$$

$$\nabla \cdot E = 0$$

use these

$$-\nabla \times (\nabla \times E) = \frac{1}{c}\partial_t (\nabla \times B)$$

$$\frac{1}{\mu}\nabla \times B = \frac{1}{c}\varepsilon\, \partial_t E$$

$$-\left[\nabla (\cancel{\nabla \cdot \vec{E}}) - \nabla^2 \vec{E}\right] = \frac{\mu\varepsilon}{c^2}\partial_t^2 E$$

$$\nabla^2 E = \frac{\mu\varepsilon}{c^2}\partial_t^2 E$$

So substiuting, $\vec{E} = \vec{\mathcal{E}}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$, we find:

(2) $$-k^2 + \frac{\mu\varepsilon}{c^2}\omega^2 = 0, \quad \text{or} \quad \omega = \frac{ck}{n} \qquad n = \sqrt{\mu\varepsilon}.$$

Then substituting:

$$\vec{H} = \vec{\mathcal{H}} e^{ik\cdot\vec{r} - i\omega t}$$ into (1) gives

$$-i\vec{k} \times \mathcal{E} = -\frac{i\omega}{c}\mu\, \vec{\mathcal{H}}$$

② Transmission

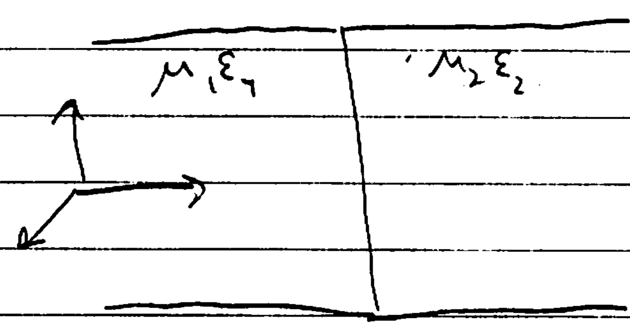
Using

$$\omega = \frac{ck}{\sqrt{\mu\varepsilon}}$$

We have

$$\frac{1}{Z}\frac{\vec{k}}{k} \times \vec{\mathcal{E}} = \vec{\mathcal{H}} \qquad \text{where} \quad Z = \sqrt{\frac{\mu}{\varepsilon}}$$

b)



Then:

$$E_{①} = \left(E_I e^{ikz} + E_R e^{-ikz}\right)\hat{x}$$

$$H_{①} = \frac{1}{Z_1}\left(E_I e^{ikz} - E_R e^{-ikz}\right)\hat{y} \quad \leftarrow \text{part a}$$

$$E_{②} = E_T e^{ikz}\,\hat{x}$$

$$H_{②} = \frac{1}{Z_2} E_2 e^{ikz}\,\hat{y} \quad \leftarrow \text{part a}$$

③ Transmission

Then $H_{\parallel}$ is continuous and $E_{\parallel}$ is continuous so:

$$E_I + E_R = E_T$$

$$\frac{1}{Z_1}(E_I - E_R) = \frac{1}{Z_2} E_T$$

Solving:

$$E_I - E_R = \frac{Z_1}{Z_2} E_T$$

$$2E_I = (1 + Z_1/Z_2) E_T$$

$$\frac{2E_I}{(1 + Z_1/Z_2)} = E_T$$

Thus,

$$\boxed{t = \frac{E_T}{E_I} = \frac{2Z_2}{(Z_1 + Z_2)}}$$

and

$$T_p = \frac{\frac{c}{Z_2} E_T^2}{\frac{c}{Z_1} E_I^2} = \frac{Z_1}{Z_2} t^2 = \boxed{\frac{2Z_1Z_2}{(Z_1 + Z_2)^2} = T_p}$$

(4) Transmission

c) For the slab we have the analagous forms:

$$E_{①} = (E_I e^{ikz} + E_R e^{-ikz})\ \hat{x}$$

$$E_{②} = (\tilde{E}_I e^{i\tilde{k}z} + \tilde{E}_R e^{-i\tilde{k}z})\ \hat{x} \leftarrow \tilde{k} = nk = \text{wave \# in medium}$$

$$E_{③} = (E_T e^{ik(z-d)} e^{i\tilde{k}d})\ \hat{x} \leftarrow \text{extra phase for convenience}$$

$$H_{①} = (E_I e^{ikz} - E_R e^{-ikz})\ \hat{y} \leftarrow \text{use part (a) with } Z=1$$

$$H_{②} = \frac{1}{Z}(\tilde{E}_I e^{i\tilde{k}z} - \tilde{E}_R e^{-i\tilde{k}z})\ \hat{y}$$

$$H_3 = (E_T e^{ik(z-d)} e^{i\tilde{k}d})\ \hat{y}$$

Continuity of E and H at $z=0$ gives

(1) $$E_I + E_R = \tilde{E}_I + \tilde{E}_R$$

(2) $$E_I - E_R = \frac{1}{Z}(\tilde{E}_I - \tilde{E}_R)$$

And at $z=d$

(3) $$e^{i\tilde{k}d}(\tilde{E}_I + \tilde{E}_R e^{-2i\tilde{k}d}) = E_T e^{i\tilde{k}d}$$

(4) $$\frac{e^{i\tilde{k}d}}{Z}(\tilde{E}_I - \tilde{E}_R e^{-2i\tilde{k}d}) = E_T e^{i\tilde{k}d}$$

Eqs (1), (2), (3), (4) are sufficient for solving for $E_R, \tilde{E}_I, \tilde{E}_R, E$

(5) Transmission

d) Then $T_p = \left| \dfrac{4Z}{(1+Z)^2 - (Z-1)^2 e^{2i\tilde{k}d}} \right|^2$ ($\star$)

Then we note that for $\tilde{k}d = n\pi$ we have a maxima, and we have a minima at $\tilde{k}d = n\pi/2$

The maxima have values:

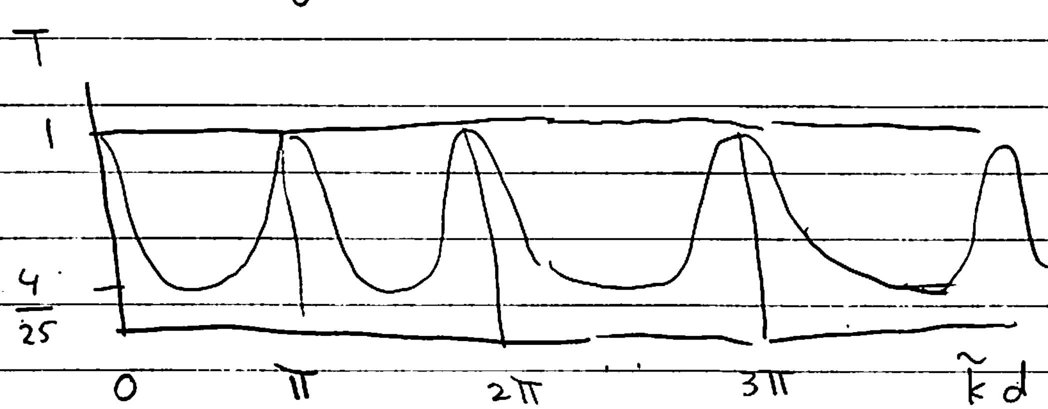
$$\frac{4Z}{(1+Z)^2 - (Z-1)^2} = 1$$

The minima have:

$$\frac{4Z}{(1+Z)^2 + (Z-1)^2} \simeq \frac{4Z}{2Z^2} \quad \text{for large } Z$$

$$\simeq \frac{2}{5} \quad \text{for } Z = 5$$

So for a pure plane wave, we have the following picture



⑥ Transmission

Now consider a wave packet

$$\Delta x \Delta k = 1/2 \qquad \Delta k = 1/2\Delta x$$

- After passsing into the dielectric, $k \to \tilde{k} = nk$. So the width of the wave packet, $\Delta k \to \Delta\tilde{k} = n\Delta k$, changes. $(\Delta\tilde{k})$

- When there is a finite width the different fourier coefficients will not add coherently when $\Delta\tilde{k}\, d \sim 1$. At this point, the response will be maximum $(\tilde{k} d = n\pi)$, for some parts of the wave packet, but a minimum $((\tilde{k} + \Delta k) d = (n + \frac{1}{2})\pi)$ for others and the interefe rence structure will wash out:

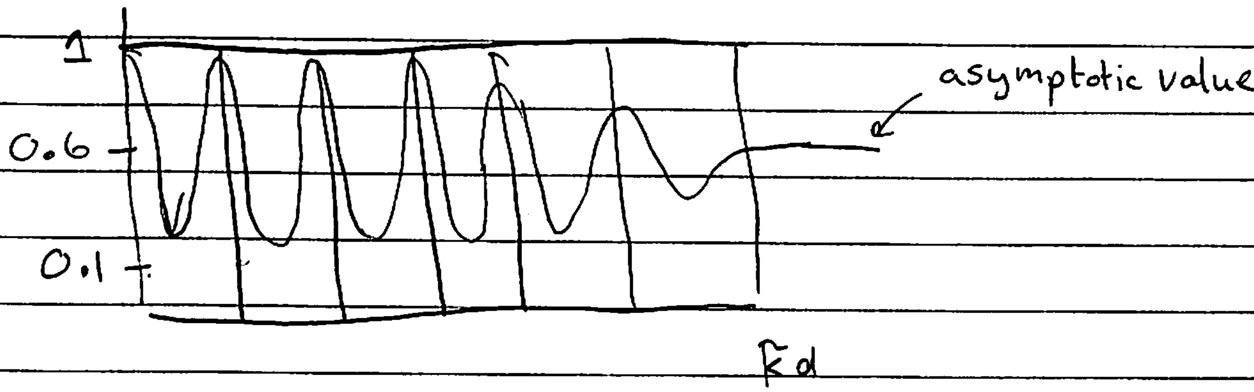
$$\Delta\tilde{k}\, d = \frac{\Delta\tilde{k}}{\bar{\tilde{k}}}(\tilde{k} d) = \frac{\Delta k}{\bar{k}}(\tilde{k} d) \sim \underbrace{\frac{1}{2\Delta x\, \bar{k}}}_{\sim 1/20} \bar{\tilde{k}} d$$

Thus, $\Delta\tilde{k}\, d \sim \frac{1}{20}\, \bar{\tilde{k}} d \sim 1$, for

$\bar{\tilde{k}} d$ about $20 \sim 6\pi$. So we expect to see about six fringes before the interference struture washes out.

⑦ Transmission

So we expect the following picture:



where the asymptotic value is given by the non-oscillating part of Eq ★ (see pg. 5)

$$T \longrightarrow \left| \frac{4Z}{(1+Z)^2} \right|^2$$

The asymptotic value is given by the product of transmission amplitudes squared:

$$T_p = |t_1 t_2|^2 = \left| \frac{4Z}{(1+Z)^2} \right|^2 \simeq \left(\frac{4Z}{Z^2} \right)^2 \sim 0.6$$

where, from (b):

t_1 = vacuum to dielectric amplitude $= 2Z/(1+Z)$

t_2 = dielectric to vacuum amplitud $= \dfrac{2}{(1+Z)}$