Grad, Div, Curl, and Laplacian

CARTESIAN $\quad d\mathbf{l} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad d^3r = dxdydz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $\quad d\mathbf{l} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}} \qquad d^3r = \rho d\rho d\phi dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $\quad d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \qquad d^3r = r^2\sin\theta drd\theta d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in Cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\, \nabla \cdot \mathbf{A} = \int_S dS\, \hat{\mathbf{n}} \cdot \mathbf{A}$$

$$\int_V d^3r\, \nabla \psi = \int_S dS\, \hat{\mathbf{n}} \psi$$

$$\int_V d^3r\, \nabla \times \mathbf{A} = \int_S dS\, \hat{\mathbf{n}} \times \mathbf{A}$$

$$\int_S dS\, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_C d\boldsymbol{\ell} \cdot \mathbf{A}$$

$$\int_S dS\, \hat{\mathbf{n}} \times \nabla \psi = \oint_C d\boldsymbol{\ell}\, \psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and $\mathbf{A}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector fields.

Problem 1. A line of charge and a wedge

(a) Determine the potential from a line of charge with uniform charge per length λ in Heavyside-Lorentz units.

(b) Consider a line charge (with charge per length λ) parallel to the z axis, but displaced from a grounded plane by a distance ρ_o (see figure). Determine the potential $\varphi(\rho, \phi)$ of the line of charge far from the plane, $\rho \gg \rho_o$. (Here $\rho = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}(y/x)$.)

(c) Now, in the same setup as part (b), consider a point-like electric dipole $\boldsymbol{p} = p_o\,\hat{\boldsymbol{y}}$ at a distance ρ on the x axis, far from the line and the plane (see figure).

  (i) What is the direction of the force on the dipole? Explain.

  (ii) Determine the force on the dipole.

(d) Now consider a grounded wedge (with opening angle ϕ between $-\beta$ and β) and line of charge with charge per length λ located on the x-axis a distance ρ_o from the origin (see below)

  (i) Using the separation of variables technique show that the separated solutions to the homogeneous Laplace equation (i.e. without the line of charge) take the form

$$\varphi_m(\rho, \phi) = \left(A_m \rho^m + \frac{B_m}{\rho^m}\right) \Phi_m(\phi) \qquad \text{where} \qquad \Phi_m(\phi) = \begin{cases} \cos(m\phi) \\ \sin(m\phi) \end{cases} \tag{1}$$

  Determine the allowed values of m and whether a specific value of m is associated with the cos or sin function.

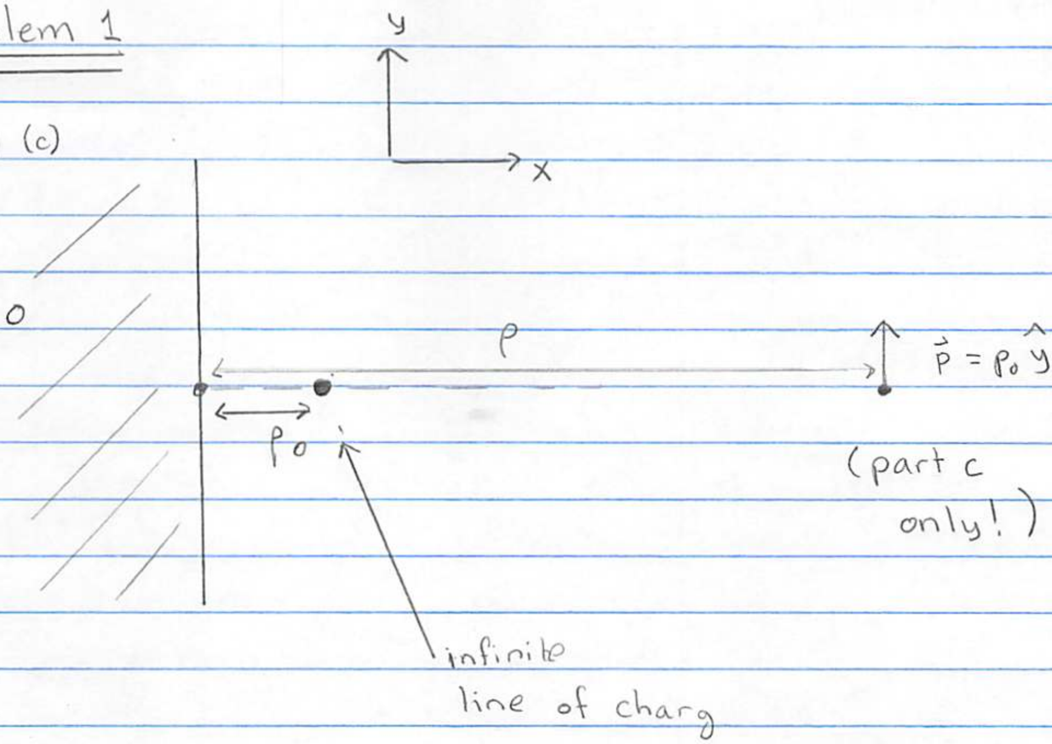
  (ii) The functions $\Phi_m(\phi)$ are a complete set of the eigen-functions for the space of functions satisfying the boundary conditions. What is the completeness relation for these eigen-functions?

  (iii) Now introduce the line of charge, and express the potential from the line of charge as a expansion homogeneous solutions, $\varphi_m(\rho, \phi)$.

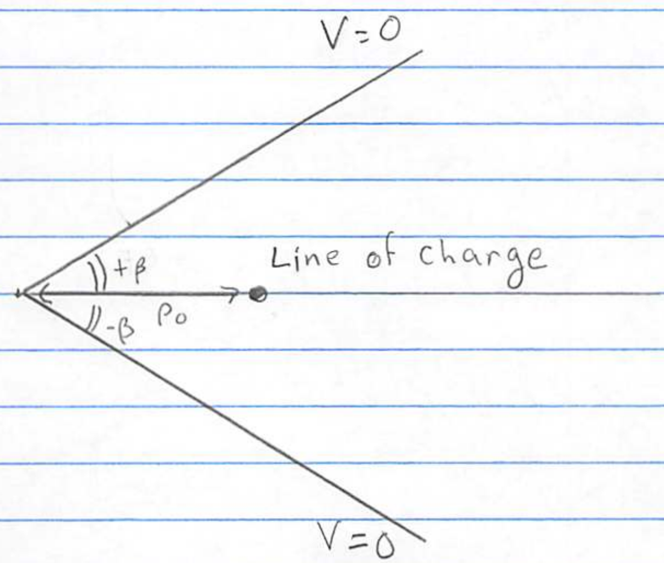
  (iv) What is the asymptotic form of the potential far from the line of charge $\rho \to \infty$. Check that this asymptotic form agrees with part (b) for the appropriate value of β.

Problem 1

Parts (b) (c)



Part (d)



Problem 2.   A dipole and a ring

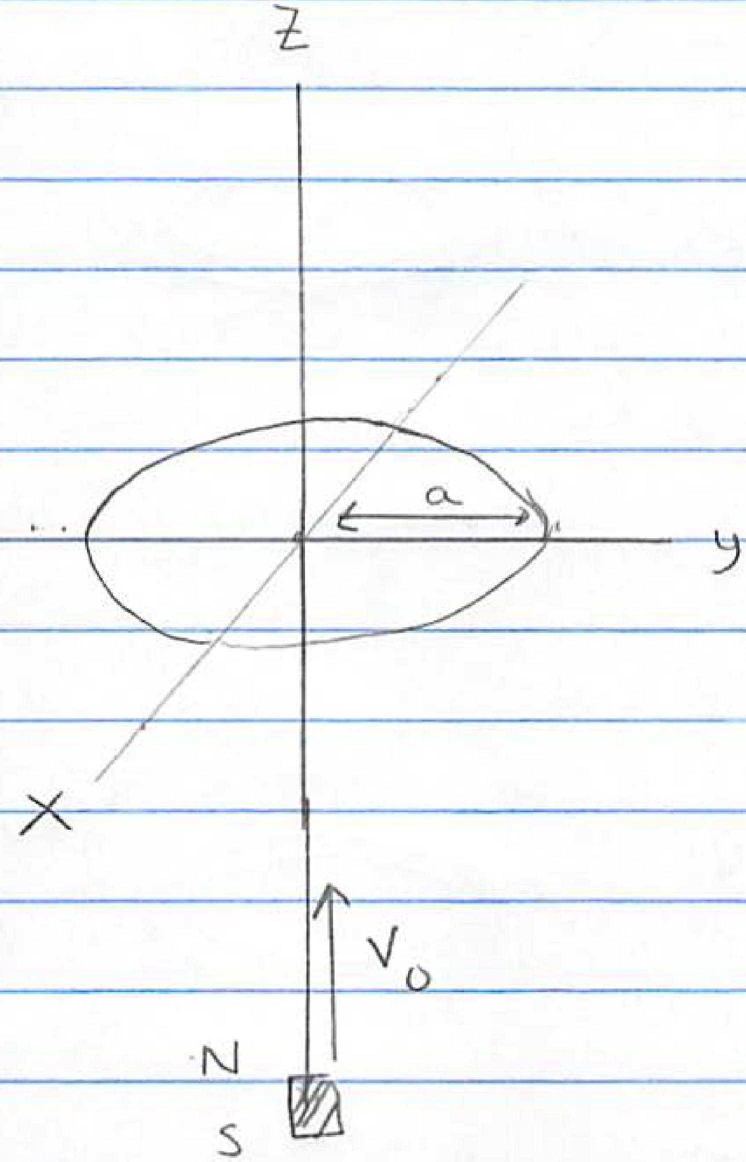
A magnetic dipole of magnetic moment m moves at constant velocity v_o up the z-axis. At time $t = 0$ the magnet moves a through a circular loop of wire of radius a, which is lying flat in the $x - y$ plane and is centered at the origin (see below). The loop to has self inductance L, but negligible resistance, $\sigma = \infty$.

(a) What is the total magnetic flux through the loop at all times. Explain. (*Hint:* Consider what would happen if the magnetic flux were to change.)

(b) Determine flux due to the magnetic dipole as a function of time.

(c) Determine the current in the loop as a function of time. Indicate whether a positive current means clockwise or counter-clockwise when viewed from above.

Problem 2



Problem 3. The magnetic field in a conducting tube

An infinitely long cylindrical shell of radius a has conductivity σ and thickness $h \ll a$. Inside and outside the shell is free space.

(a) If the magnetic field inside the tube is uniform and varies in time as $\boldsymbol{B}(t, \boldsymbol{x}) = B_{\rm in} e^{-i\omega t}\,\hat{\boldsymbol{z}}$, determine:

  (i) The vector potential inside the tube.

  (ii) The electric field inside the tube.

(b) Now consider the same tube in the presence of a homogeneous external applied field $\boldsymbol{B}_{\rm ext}(t, \boldsymbol{x}) = B_o e^{-i\omega t}\,\hat{\boldsymbol{z}}$ outside of the tube. The goal of this problem is to determine the magnetic field inside the tube. Make no assumptions about the skin depth compared to thickness h.

  (i) Show quite generally that inside the metal the magnetic field obeys the diffusion equation

$$\partial_t \boldsymbol{B}(t, \boldsymbol{x}) = D \nabla^2 \boldsymbol{B}(t, \boldsymbol{x})\,. \tag{2}$$

  and determine the diffusion coefficient.

  (ii) Look for plane wave solutions to the diffusion equation in one dimension $B(x, t) = B_o e^{-i\omega t + ikx}\,\hat{\boldsymbol{z}}$. Solve for the allowed values of k as a function of frequency.

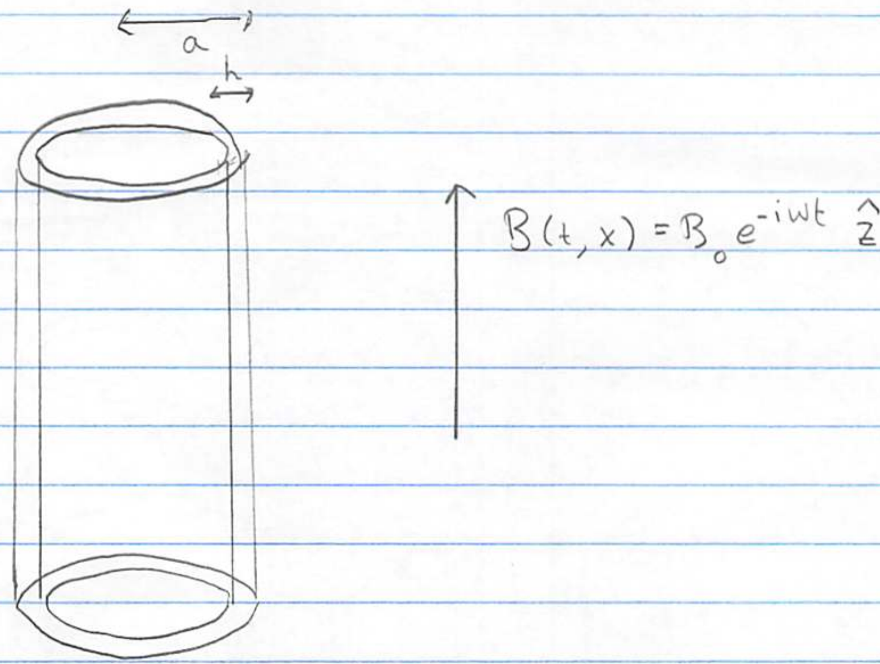
  (iii) Set up a system of linear equations which determine the magnetic inside the cylinder. Explicitly check that you have enough independent equations to determine all unknowns.

  *H*int: Treat the magnetic field inside and outside the cylinder as uniform. Then relate the inside and outside by solving for the fields across the interface. Treat the interface as an infinite plane as shown below.

  (iv) (Time permitting) Determine the magnitude of the magnetic field in the cylinder in terms of B_o in the limit that $h/\delta \gg 1$ and $\delta/a \ll 1$.

Problem 3

Parts (a) and (b)



Part (b) iii - magnification of shell wall.