Bohr

a) $a_0 = \dfrac{\hbar}{mc\alpha}$

$$E = -\frac{1}{2}\frac{e^2}{4\pi a_0}$$

b) Using the Larmour formula

$$a = \omega_0^2 a_0$$

$$P = \frac{e^2}{4\pi}\frac{2}{3}\frac{a^2}{c^3}$$

☆ $$\boxed{P = \frac{e^2}{4\pi}\frac{2}{3}\frac{\omega_0^4 a_0^2}{c^3}}$$ $\omega_0 = \dfrac{\alpha c}{a_0}$

$$P = \frac{e^2}{4\pi}\frac{2}{3}\frac{\alpha^4 c}{a_0^2}$$

☆☆ $$\boxed{P = \frac{e^2}{4\pi a_0}\frac{2}{3}\alpha^4\frac{c}{a_0}}$$

c) So for part

$$\Delta E = \frac{2\pi a_0}{\alpha c} P$$

So
$$\Delta E = \frac{e^2}{4\pi a_0} \frac{2}{3} \alpha^4 \frac{c}{a_0} \frac{2\pi a_0}{c\alpha}$$

$$\Delta E = \frac{e^2}{4\pi a_0} \frac{4\pi}{3} \alpha^3$$

And
$$\boxed{\frac{\Delta E}{E} = \frac{8\pi}{3}\alpha^3} \sim 10^{-6}$$

d) Using the Larmour result

$$E_{rad} = \frac{e}{4\pi r c^2} n \times n \times a(t_e)$$

Using
$$\vec{r} = (\cos \omega_0 t_e, \sin\omega_0 t_e, 0)\, a_0$$

$$a = -\omega_0^2 a_0 (\cos\omega_0 t, \sin\omega_0 t, 0)$$

We will take the real part of this

$$\Rightarrow \vec{a} = -\omega_0^2 a_0 (\hat{x} e^{-i\omega_0 t} + i\hat{y} e^{-i\omega_0 t})$$

$$\vec{a} = -\omega_0^2 a_0 e^{-i\omega t} (\hat{x} + i\hat{y})$$

So then

$$\frac{d\bar{P}}{d\Omega} = \frac{e^2}{16\pi c^3} \left|\frac{\vec{n}\times\vec{n}\times\vec{a}}{2}\right|^2$$

$$n\times n\times\vec{a} = -\vec{a} + \vec{n}\,(n\cdot\vec{a})$$

So taking $\vec{n}$ in the x, z plane:

$$n\times n\times\vec{a} \propto -(\hat{x}+i\hat{y}) + \vec{n}\,(\vec{n}\cdot(\hat{x}+i\hat{y}))$$

$$\propto -(\hat{x}+i\hat{y}) + \vec{n}\,(n\cdot\hat{x})$$

$$|n\times n\times\vec{a}|^2 \propto (\hat{x}^2+\hat{y}^2) - (n\cdot x)^2$$

$$\propto 2 - \sin^2\theta$$

Then

$$\frac{d\bar{P}}{d\Omega} = \frac{e^2}{16\pi c^3}\,\frac{1}{2}\,(2-\sin^2\theta)\,(\omega_0^2 a_0)^2$$

e) To check this result we integrate over Ω

$$\bar{P} = \frac{e^2}{16\pi c^3}\left(\frac{\omega_0^2 a_0}{2}\right)^2 \underbrace{\int d\Omega\,(2-\sin^2\theta)}_{I}$$

Evaluating I:

$$I = 2\pi\int_{-1}^{1} dx\,\left(2-(1-x^2)\right) \qquad x \equiv \cos\theta$$

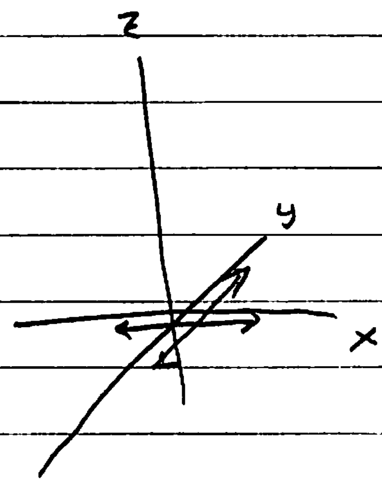
$$= 2\pi\int_{-1}^{1} dx\,(1+x^2)$$

$$I = 2\pi\left(2+\frac{2}{3}\right) = \frac{16\pi}{3}$$

So $$\bar{P} = \frac{e^2}{16\pi c^3}\left(\frac{\omega_0^2 a_0}{2}\right)^2 \cdot \frac{16\pi}{3}$$

$$\bar{P} = \frac{e^2}{6\pi c^3}\left(\omega_0^2 a_0\right)^2$$ which agrees with ★ from part b

f) One can understand circular motion as a super-position of an x-oriented dipole and y-oriented dipole, 90° out of phase. On the x-axis only the radiation from the y-oriented dipole contributes as the x-oriented dipole is parallel to the observation direction (only transverse currents contribute to the radiation). On the z-axis both the x-oriented dipole + the y-oriented dipole contribute to the radiation field.



The two dipoles add incoherently.

Scattering

a) The incoming wave induces a dipole moment and the dipole Radiates

$$p = 4\pi a^3 (\varepsilon-1)/(\varepsilon+2)\, E_0\, e^{-i\omega t + ikz}$$

Then using formulas for dipole radiation

$$\overline{P} = \frac{1}{4\pi c^3} \frac{\omega^4 |p_0|^2}{3}$$ ← time averaged power due to dipole radiation

So we have with

$$p_0 = 4\pi a^3 (\varepsilon-1)/(\varepsilon+2)\; E_0$$

$$\overline{P} = \frac{4\pi \omega^4}{3} a^6 \left(\frac{\varepsilon-1}{\varepsilon+2}\right)^2 E_0^2$$

And the cross section

$$\sigma = \frac{\overline{P}}{\frac{1}{2} c |E_0|^2} = \frac{\text{Time averaged Power out}}{\text{Time ave Power in}}$$

$$\sigma = \frac{8\pi}{3}\left(\frac{\omega a}{c}\right)^4 a^2$$

Scattering pg. 2

b) Using

$$\vec{E}_{rad} = \frac{1}{4\pi r c^2} \; n \times n \times \ddot{p}(t_e)$$

We use

$p(t_e) = p_0 e^{-i\omega t_e}$

$$\vec{E}_{rad} = -\frac{\omega^2}{4\pi r c^2}\left[-\vec{p}_0 + \vec{n}(n\cdot\vec{p}_0)\right] e^{-i\omega t_e}$$

$$\frac{d\overline{P}}{d\Omega} = \frac{c}{2}\left|r E_{rad}\right|^2 = \frac{\omega^4}{16\pi^2 c^3}\left(\vec{p}_0^{\,2} - (n\cdot\vec{p}_0)^2\right)\frac{1}{2}$$

time average

time average

With the incoming light we have the induced moment

$$\vec{E} = E_0 \vec{\mathcal{E}}_0 e^{-i\omega t + ikz}, \quad \vec{p} = \alpha_E \vec{E}, \quad \alpha_E \equiv 4\pi a^3\left(\frac{\varepsilon-1}{\varepsilon+2}\right)$$

Then:

$$\frac{d\overline{P}}{d\Omega} = \frac{c E_0^2}{2}\frac{\omega^4}{16\pi^2 c^4}\left(\mathcal{E}_0^2 - (n\cdot\mathcal{E}_0)^2\right)\alpha_E^2$$

Using:

$$\mathcal{E}_0 = (1, 0, 0)$$

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{n}\cdot\mathcal{E}_0 = \sin\theta\cos\phi$$

and

$$\frac{d\overline{P}}{d\Omega} = \frac{cE_0^2}{2} \frac{\omega^4}{16\pi^2 c^4} \left(1 - \sin^2\theta\cos^2\phi\right) \alpha_E^2$$

And then as before

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\overline{dP/d\Omega}}{\frac{c}{2} E_0^2} = a^2 \cdot \left(\frac{\omega a}{c}\right)^4 \left(1 - \sin^2\theta\cos^2\phi\right) \left(\frac{\varepsilon - 1}{\varepsilon + 2}\right)^2}$$

So to quickly check that this is consistent with part a) we integrate over the solid angle:

$$I = \int d\Omega \left(1 - \sin^2\theta \cos^2\phi\right)$$

we integrate over ϕ in this step, and then set $x = \cos\theta$

$$I = 2\pi \int_{-1}^{1} dx \left(1 - \sin^2\theta \, \frac{1}{2}\right)$$

$$= 2\pi \int_{-1}^{1} dx \left(1 - (1 - x^2)\frac{1}{2}\right)$$

$$= 2\pi \int_{-1}^{1} dx \, \frac{1}{2} + \frac{x^2}{2}$$

$$= 2\pi \left[1 + \frac{1}{3}\right] = 8\pi/3$$

So we see again that

$$\sigma = \frac{8\pi}{3}\left(\frac{\omega a}{c}\right)^4 a^2$$

c) Now consider two spheres the two spheres have the induced moment

$z = 0$ $\qquad\qquad$ $z = b$

$$p_0 = \alpha_E \vec{\varepsilon}_0 \, E_0 \, e^{-i\omega t} \, e^{ik0}$$

$$p_b = \alpha_E \vec{\varepsilon}_0 \, E_0 \, e^{-i\omega t} \, e^{ikb}$$

So the radiation field is

$$E_{rad} = \frac{-\omega^2}{4\pi r c^2}\left(-p_0 + \vec{n}(\vec{n}\cdot\vec{p}_0)\right) e^{-i\omega t_e}$$

$$+ \frac{-\omega^2}{4\pi r c^2}\left(-p_0 + \vec{n}(\vec{n}\cdot\vec{p}_0)\right) e^{-i\omega t_e} e^{ikb}$$

$$E_{rad} = \frac{-\omega^2}{4\pi r c^2} e^{-i\omega t_e}\left(-p_0 + \vec{n}(\vec{n}\cdot p_0)\right)\left(1 + e^{ikb}\right)$$

Comparison with ~~A~~ of part b)

We see that E_{rad} is multiplied by a factor

$$(1 + e^{ikb})$$

Then $|E_{rad}|^2$ is multiplied by the square of this factor

$$\frac{dP}{d\Omega} = \frac{cE_0^2}{2} \frac{\omega^4}{16\pi^2 c^4} (1 - \sin^2\theta \cos^2\phi)\, \alpha_E^2 \, |1 + e^{ikb}|^2$$

Radiation during deceleration

a) $$\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}} \simeq \frac{1}{\sqrt{2(1-\beta_0)}}$$

b) We use the following result

$$E_{rad} = \frac{q}{4\pi r c^2} \frac{n \times (n-\beta) \times a}{(1-n\cdot\beta)^3} = \frac{q}{4\pi r c^2} \frac{\vec{n} \times \vec{n} \times \vec{a}}{(1-n\cdot\beta)^3}$$

$\vec{\beta}$ is // to $\vec{a}$

So

$$\frac{dW}{dT\,d\Omega} = c\,\vec{E}_{rad}^{\,2} \,\frac{dt}{dT}$$

$$= \frac{q^2}{16\pi^2 c^3}\left(\frac{\vec{n}\times\vec{n}\times\vec{a}}{(1-n\cdot\beta)^3}\right)^2 (1-n\cdot\beta)$$

$$\star \quad \boxed{\frac{dW}{dT\,d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{(n\times n\times a)^2}{(1-n\cdot\beta)^5}}$$

We should probably emphasise that β is a function of time in Eq $\star$:

$$\beta(T) = \beta_0\left(1 - \frac{T}{\Delta t}\right)$$

Taking a specific axis

$$(n \times n \times \vec{a})^2 = a_T^2 = a^2 \sin^2\theta$$

Then

$$\frac{dW}{dT d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{a^2 \sin^2\theta}{(1 - \beta_0(1 - T/\Delta t)\cos\theta)^5}$$

Where $a = \beta_0 c/\Delta t$.

c) Then we approximate this result. When $\beta \simeq 1$, $\theta \simeq 0$, and $T \simeq 0$ the denominator,

$$(1 - \beta_0(1 - T/\Delta t)\cos\theta)^5 \longrightarrow 0^5$$

Vanishes very fast, making the energy strongly enhanced.

$$1 - \beta_0 \cos\theta + \frac{\beta_0 T}{\Delta t} \cos\theta = (1 - \vec{n} \cdot \vec{\beta}(T))$$

Then we write

$$\beta_0 = 1 - \underbrace{\delta\beta_0}_{\text{small}} \qquad T \simeq \text{small}$$

$$\cos\theta \simeq 1 + \theta^2/2$$

So:

$$1 - \vec{n}\cdot\vec{\beta}(T) = 1 - (1-\delta\beta_0)(1+\theta^2/2) + \frac{T}{\Delta t}\frac{(1-\delta\beta)}{(1+\theta^2/2}$$

Keeping terms first order in smallness

$$1 - \vec{n}\cdot\vec{\beta} = \delta\beta_0 + \frac{\theta^2}{2} + \frac{T}{\Delta t}$$

$$= \frac{1}{2\gamma_0^2} + \frac{\theta^2}{2} + \frac{T}{\Delta t}$$

Now

$$\frac{dW}{dT\,d\Omega} = \frac{q^2}{16\pi^2 c^3}\frac{a^2 \sin^2\theta}{\left(\frac{1}{2\gamma_0^2} + \frac{\theta^2}{2} + \frac{T}{\Delta t}\right)^5}$$

$$= \frac{q^2}{16\pi^2 c^3}\frac{2^5\,\gamma_0^{10}\,\theta^2}{(1+(\gamma_0\theta)^2 + \gamma_0^2 T/\Delta t)^5}$$

i + ii) we see that the function (the energy) is a function of $(\gamma_0\theta)$ and $\gamma_0^2 T/\Delta t$

Thus...

The value of the energy will change for

$$\gamma\theta \sim 1 \quad \text{and} \quad \gamma_0^2 T/\Delta t \sim 1$$

Thus

i) $\theta \sim \dfrac{1}{\gamma} \sim 10^{-4}$ rad

ii) $T \sim \dfrac{\Delta t}{\gamma_0^2} \sim 10^{-8}$ s

d) Now to determine $dW/d\Omega$ we integrate over T.

$$\frac{dW}{d\Omega} = \int_0^\infty dT \, \frac{dW}{dT\, d\Omega}$$

$$= \int_0^{\varepsilon} \frac{q^2}{16\pi^2 c^3} \, 2^5 \, \gamma_0^8 \, \frac{(\gamma_0\theta)^2}{(1 + (\gamma_0\theta)^2 + \gamma_0^2 T/\Delta t)^5} \, dT$$

$$= \frac{2q^2}{\pi^2 c^3} \cdot \gamma_0^8 \left[\int_0^{\varepsilon} dT \, \frac{(\gamma_0\theta)^2}{(1 + (\gamma_0\theta)^2 + \gamma_0^2 T/\Delta t)^5} \right]$$

$$\frac{dW}{d\Omega} = \frac{2q^2}{\pi^2 c^3} \, \gamma_0^8 \, \frac{\Delta t}{\gamma_0^2} \int_0^{\gamma_0^2 \varepsilon/\Delta t} du \, \frac{(\gamma_0\theta)^2}{(1 + (\gamma_0\theta)^2 + u)^5}$$

where $u = \gamma_0^2 T/\Delta t$. Extending the integral up to infinity

$$I = \int_0^\infty du \frac{(\gamma_0\theta)^2}{(1+(\gamma_0\theta)^2+u)^5}$$

$$= -\frac{1}{4}\frac{(\gamma_0\theta)^2}{(1+(\gamma_0\theta)^2+u)^4}\Bigg|_0^\infty$$

$$I = \frac{1}{4}\frac{(\gamma_0\theta)^2}{(1+(\gamma_0\theta)^2)}$$

So

$$\frac{dW}{d\Omega} = \frac{2q^2}{\pi^2 c^3}\gamma_0^6 \Delta t \frac{1}{4}\frac{(\gamma_0\theta)^2}{(1+(\gamma_0\theta)^2)^4}$$

e) The time scale for the emission (formation time)

$$\Delta T \sim \frac{\Delta t}{\gamma_0^2}$$

The observation time is

$$\Delta t_{obs} = \Delta T \frac{\Delta t_{obs}}{\Delta T} = \Delta T \overbrace{(1 - n\cdot\beta)}^{\sim 1/\gamma_0^2}$$

$$\sim \frac{\Delta t}{\gamma_0^2}\frac{1}{\gamma_0^2} \sim \Delta t/\gamma_0^4$$

So the typical frequency that is emitted is

$$\omega_{typ} \sim \frac{1}{\Delta t_{obs}} \sim \frac{\gamma_0^4}{\Delta t} \sim (10^4)^4 \frac{1}{s}$$

$$\omega_{typ} \sim 10^{16} \frac{1}{s} \sim \text{optical frequencies}$$

Changing Frames

According to the Lorentz Transformation

$$\underline{F}^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} F^{\rho\sigma}$$

Since $F^{\rho\sigma} = 0$ for $\rho,\sigma \neq 0,i$ or $i,0$ we have

$$\underline{F}^{\mu\nu} = L^{\mu}{}_{0} L^{\nu}{}_{i} F^{0i} + L^{\mu}{}_{i} L^{\nu}{}_{0} F^{i0}$$

Take

$$\underline{F}^{0x} = L^{0}{}_{0} L^{x}{}_{x} F^{0x} + L^{0}{}_{i} L^{x}{}_{0} F^{i0}$$

With:

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} L^{0}{}_{0} & L^{0}{}_{1} \\ L^{1}{}_{0} & L^{1}{}_{1} \end{pmatrix}$$

$$\underline{E}^{x} = \underline{F}^{0x} = \gamma^2 E^x + (-\gamma\beta)(-\gamma\beta)(-E^x)$$

$$= \gamma^2 E^x (1 - \beta^2)$$

$$= E^x$$

Changing Frames 2

Similarly

$$\underline{F}^{0y} = L^0{}_0 \, L^y{}_i \, F^{0i} + L^0{}_i \, \cancelto{0}{L^y{}_0} \, F^{i0}$$

$$\boxed{\underline{F}^{0y} = (\gamma)(1)\, E^y}$$

Then

$$\underline{F}^{xy} = L^x{}_0 \, L^y{}_i \, F^{0i} + L^x{}_i \, L^y{}_0 \, F^{i0}$$

$$\underline{F}^{xy} = (-\gamma\beta)(1)\, E^y$$

$$\boxed{\underline{B}^z = \underline{F}^{xy} = -\gamma\beta\, E^y}$$

The remaing $F^{\mu\nu}$ components are zero, e.g.

$$\underline{F}^{\mu\nu} = L^\mu{}_0 \, L^\nu{}_i \, F^{0i} + L^\mu{}_i \, L^\nu{}_0 \, F^{i0}$$

If μ, ν do not contain x

$$\underline{F}^{yz} = L^y{}_0 \, L^z{}_i \, \cancelto{0}{F^{0i}} + L^y{}_i \, \cancelto{0}{L^z{}_0} \, F^{i0} = 0$$

$$\underline{F}^{xz} = L^x{}_0 \, L^z{}_i \, \cancelto{0}{F^{0i}} + L^x{}_i \, \cancelto{0}{L^z{}_0} \, F^{i0} = 0$$

Changing Frames pg. 3

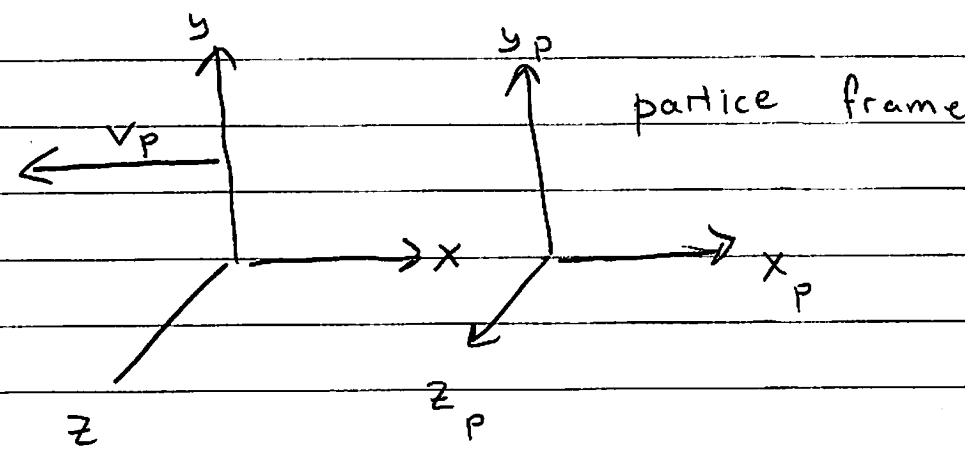
b) We know the fields in the frame of the particle. The Coulomb Law gives

$$E^x = \frac{q}{4\pi r_p^2}\hat{r}_p^{\,x} = \frac{q}{4\pi}\frac{x_p}{(x_p^2+y_p^2)^{3/2}}$$

$$E^y = \frac{q}{4\pi r_p^2}\hat{r}_p^{\,y} = \frac{q}{4\pi}\frac{y_p}{(x_p^2+y_p^2)^{3/2}}$$

We notate the particles coords (t_p, x_p, y_p, z_p)

Then in the frame of Lab



We need to boost these fields, using the Lorentz transformation rules. The Lab frame is moving to the left relative to the particle

$$\begin{pmatrix} ct \\ x \\ y \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 \\ \gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct_p \\ x_p \\ y_p \end{pmatrix}$$

Changing Frames pg. 4

So

$$ct = \gamma c t_p + \gamma \beta x_p$$

and $y = y_p$

$$x = \gamma \beta c t_p + \gamma x_p$$

So multiplying the first eqn by β and subtractin

$$x - v_p t = \gamma x_p - \gamma \beta^2 x_p$$

$$x - v_p t = \frac{x_p}{\gamma}$$

$$\gamma (x - v_p t) = x_p$$

Then we substitute into the transformation rules

$$\underline{E}^x = E^x = \frac{q}{4\pi} \frac{x_p}{(x_p^2 + y_p^2)^{3/2}}$$

$$\boxed{\underline{E}^x = \frac{q}{4\pi} \frac{\gamma (x - v_p t)}{(\gamma^2 (x - v_p t)^2 + y^2)^{3/2}}}$$

$$\underline{E}^y = \gamma E^y = \boxed{\frac{q}{4\pi} \frac{\gamma y}{(\gamma^2 (x - v_p t)^2 + y^2)^{3/2}} = \underline{E}^y}$$

Finally we evaluate B^z

$$\underline{B}^z = +\beta\gamma E^y$$

$$= +V_p \gamma E^y$$

note we have changed the sign of β relative to part a since in part (a) the new observer was moving to the right. Now the new observer (the person sitting in the Lab) is moving to the left.

$$\boxed{\underline{B}^z = V_p \underline{E}^y}$$

c) Then finally we make a graph

Setting $x = 0$ $y = y_0$

$$\underline{F}^x = Q\underline{E}^x = \frac{Qq}{4\pi} \frac{-\gamma ct}{((\gamma ct)^2 + y_0^2)^{3/2}}$$

$$\underline{F}^y = Q\underline{E}^y = \frac{Qq}{4\pi} \frac{\gamma y_0}{((\gamma ct)^2 + y_0^2)^{3/2}}$$

Changing Frames pg. 6

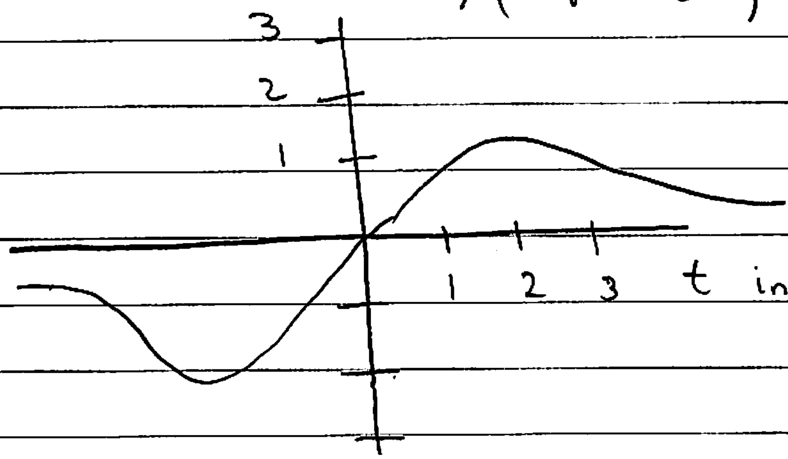
Pulling out y_0

$$F^x = \frac{Qq}{4\pi y_0^2} \; \frac{-\gamma c/y_0 \, t}{\left((\gamma c t/y_0)^2 + 1 \right)^{3/2}}$$

$$F^y = \frac{Qq}{4\pi y_0^2} \; \frac{\gamma}{\left((\gamma c t/y_0)^2 + 1 \right)^{3/2}}$$

Then plotting:

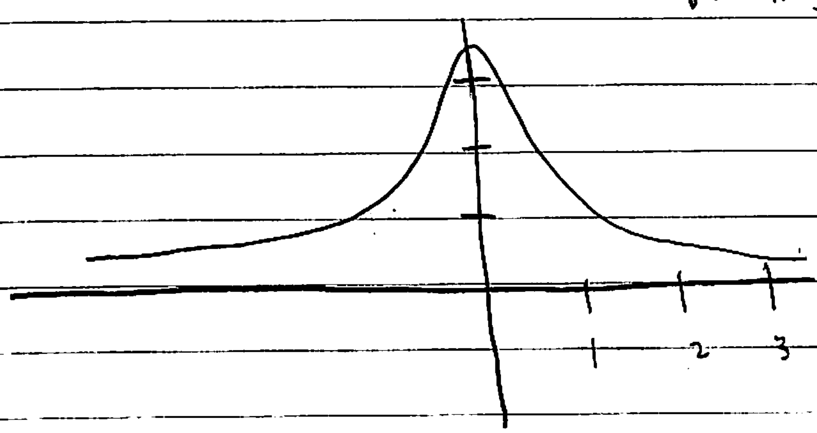
$F^x/(Qq/4\pi y_0^2)$ in units of order 1



t in units $\frac{y_0}{c\gamma} \simeq \frac{1m}{3\times10^8 m/s} \frac{1}{10^6}$

$\simeq 10^{-14} s$

$F^y/(Qq/4\pi y_0^2)$ in units 10^6



t in units $10^{-14} s$