1 Retarded Green Functions and Green Theorem

1.1 Linear operators, boundary conditions, and adjoints

Consider the simplest of all differential operator

$$\mathcal{L}_t = \left[\frac{d}{dt}\right] \tag{1.1}$$

The operator is fully specified by its action *and* the space of functions on which it acts, *i.e.* the boundary conditions and domain of the functions which we are talking about. For instance lets take the space of functions between $t_{\rm min}$ and $t_{\rm max}$ with retarded boundary conditions, *i.e.* $v(t) = 0$ for $t \le t_{\rm min}$.

$$\mathcal{L}_t v(t) = \left[\frac{d}{dt}\right] v(t) \qquad v(t_{\rm min}) = 0 \tag{1.2}$$

Numerically we discretize using backward differences

$$\frac{dv}{dt} = \frac{v(t) - v(t-\Delta t)}{\Delta t} \tag{1.3}$$

Then the action of the differential operator can be put in matrix form

$$\mathcal{L}_t v(t) = \begin{pmatrix} 1/\Delta t & 0 & & & \\ -1/\Delta t & 1/\Delta t & & & \\ & -1/\Delta t & \ddots & & \\ & & \ddots & 1/\Delta t & 0 \\ & & & -1/\Delta t & 1/\Delta t \end{pmatrix} \begin{pmatrix} v(t_1) \\ v(t_2) \\ v(t_3) \\ \vdots \\ v(t_N) \end{pmatrix} \tag{1.4}$$

where $t_n = t_{\rm min} + n\,\Delta$. The last point is at $t_N = t_{\rm max} - \Delta$

Can you see how the retarded boundary conditions are used when placing the entries into the matrix – basically we said

$$\left.\frac{dv(t)}{dt}\right|_{t_1} = \frac{v(t_1)}{\Delta t} \tag{1.5}$$

which is true if $v(t_{\rm min}) = 0$.

The adjoint differential operator is a unique operator satisfying

$$\int_{t_{\rm min}}^{t_{\rm max}} dt\, V_2(t)\, (\mathcal{L}_t v_1(t)) = \int dt\, (\mathcal{L}_t^{\rm adj} V_2(t))\, v_1(t) \tag{1.6}$$

Here $v_1(t)$ satisfies retarded boundary conditions (that is $v_1(t_{\rm min}) = 0$) and $V_2(t)$ satisfies the adjoint boundary conditions, which (as we will show below) are advanced boundary conditions $V_2(t_{\rm max}) = 0$ for the particular boundary conditions satisfied by $\mathcal{L}_t$.

As you know the adjoint is found by taking the hermitian transpose

$$\mathcal{L}_t^{\rm adj} v(t) = \begin{pmatrix} 1/\Delta t & -1/\Delta t & 0 & 0 & 0 \\ 0 & 1/\Delta t & \ddots & 0 & 0 \\ 0 & 0 & \ddots & -1/\Delta t & 0 \\ 0 & 0 & 0 & 1/\Delta t & -1/\Delta t \\ 0 & 0 & 0 & 0 & 1/\Delta t \end{pmatrix} \begin{pmatrix} v(t_0) \\ v(t_1) \\ v(t_2) \\ \vdots \\ v(t_N) \end{pmatrix} \tag{1.7}$$

We see that the adjoint differential operator is

$$\mathcal{L}_t^{\rm adj} = -\frac{d}{dt} \tag{1.8}$$

This only half specifies the operator. The full specification is for the space of functions $v(t)$ between $t_{\rm min} \ldots t_{\rm max}$ which vanish at the upper end

$$\mathcal{L}_t^{\rm adj} = -\frac{d}{dt} \qquad v(t_{\rm max}) = 0 \tag{1.9}$$

This follows because the discretization implies that

$$-\left.\frac{dv(t)}{dt}\right|_{t_N} = \frac{v(t_N)}{\Delta t} \tag{1.10}$$

which is true provided $v(t_{\rm max}) = 0$.

Without appealing to the discretization, the appropriate adjoint operator, and its boundary conditions are found by taking any function $V(t)$ and integrating by parts:

$$\int_{t_{\rm min}}^{t_{\rm max}} dt\, V(t)(\frac{d}{dt}v_1(t)) = \underbrace{[V(t_{\rm max})v_1(t_{\rm max}) - V(t_{\rm min})v_1(t_{\rm min})]}_{\text{bndry terms}} + \int_{t_{\rm min}}^{t_{\rm max}} dt \left[-\frac{d}{dt}V(t)\right] v_1(t) \tag{1.11}$$

Ignoring the boundary terms the adjoint operator is

$$\mathcal{L}_t^{\rm adj} = \left[-\frac{d}{dt}\right] \tag{1.12}$$

But, the full specification also specifies that that $V(t_{\rm max}) = 0$, so that the boundary terms vanish:

$$\mathcal{L}_t^{\rm adj} = \left[-\frac{d}{dt}\right] \qquad V(t_{\rm max}) = 0. \tag{1.13}$$

We conclude with examples.

Example operators and their adjoints

(a) a particle experiencing drag with time dependent mass and drag coefficient with retarded boundary conditions

$$\mathcal{L}_t = \left[m(t)\frac{d}{dt} + m(t)\eta(t)\right] \qquad v(t_{\rm min}) = 0 \tag{1.14}$$

The adjoint is

$$\mathcal{L}_t^{\rm adj} = \left[-\frac{d}{dt}m(t) + m(t)\eta(t)\right] \qquad v(t_{\rm max}) = 0 \tag{1.15}$$

(b) The damped harmonic oscillator with time dependent mass and drag coefficient

$$\mathcal{L}_t = \left[\frac{d}{dt} m(t) \frac{d}{dt} + \frac{d}{dt} m(t)\eta(t) + m(t)\omega_o^2\right] \qquad v(t) = 0 \quad t \leq t_{\min} \tag{1.16}$$

The adjoint is

$$\mathcal{L}_t^{\mathrm{adj}} = \left[\frac{d}{dt} m(t) \frac{d}{dt} - m(t)\eta(t) \frac{d}{dt} + m(t)\omega_o^2\right] \qquad v(t) = 0 \quad \geq t_{\max} \tag{1.17}$$

(c) The Sturm-Liouville operator with homogeneous boundary conditions

$$\mathcal{L}_x = \left[-\frac{d}{dx} p(x) \frac{d}{dx} + q(x)\right] \qquad y(x_{\min}) = y(x_{\max}) = 0 \tag{1.18}$$

where $p(x) > 0$ between $x_{\min}$ and $x_{\max}$ The operator is self adjoint, $\mathcal{L}_x^{\mathrm{adj}} = \mathcal{L}_x$.

Self Test:

(a) Show using any of the methods of this section that, the adjoint of Eq. (1.16) is Eq. (1.17).

1.2 The $G_R(t, t_o)$ is adjoint respect to t_o

We now show that $G_R(t, t_o)$ when considered a function of t_o satisfies the adjoint differential equation and boundary conditions, *i.e.* since

$$\mathcal{L}_t G_R(t, t_o) = \delta(t - t_o) \tag{1.19}$$

we will show that

$$\mathcal{L}_{t_o}^{\mathrm{adj}} G_R(t, t_o) = \delta(t - t_o) \tag{1.20}$$

The proof of this goes like this. We start with

$$\int_{t_{\min}}^{t_{\max}} dt' G_R(t, t') \, \mathcal{L}_{t'} G_R(t', t'') = G_R(t, t'') \, . \tag{1.21}$$

If you think of $\mathcal{L}_{t'}$ as a matrix, then it is clear that

$$\int_{t_{\min}}^{t_{\max}} dt' (\mathcal{L}_{t'}^{\mathrm{adj}} G_R(t, t')) G_R(t', t'') = G_R(t, t'') \, . \tag{1.22}$$

Or

$$\mathcal{L}_{t'}^{\mathrm{adj}} G(t, t') = \delta(t - t') \tag{1.23}$$

More formally, we can integrate by parts picking up boundary terms and the adjoint operator

$$\text{bndry-terms} + \int_{t_{\min}}^{t_{\max}} dt' \left[\mathcal{L}_{t'}^{\mathrm{adj}} G(t, t')\right] G(t', t'') = G(t, t'') \tag{1.24}$$

The only way this can be satisfied for all t and t'' is if the boundary terms vanish (i.e. $G(t, t')$ satisfies adjoint b.c. with respect to t'), and

$$\mathcal{L}_{t'}^{\mathrm{adj}} G(t, t') = \delta(t - t'') \tag{1.25}$$

1.3 Green Theorem

Now we can prove Green theorem. For definiteness, take the drag equation with retarded boundary conditions:

$$\underbrace{\left[\frac{d}{dt} + \eta\right]}_{\mathcal{L}_t} v(t) = F(t) \tag{1.26}$$

Given an inital condition at $t = 0$, $v(t_{\min}) = v_o$, and the Green function $G(t, t_o)$ we would like to write down the general (formal) solution to the equation.

The procedure to do this is always the same – act with the adjoint operator on the Green function and integrate by parts

$$v(t) = \int_{t_{\min}}^{t_{\max}} dt_o\, v(t_o) \mathcal{L}_{t_o}^{\text{adj}} G_R(t, t_o) \tag{1.27}$$

$$= \int_{t_{\min}}^{t_{\max}} dt_o\, v(t_o) \left[-\frac{d}{dt_o} + \eta\right] G_R(t, t_o) \tag{1.28}$$

Integrating by parts we get

$$v(t) = -v(t_{\max}) G_R(t, t_{\max}) + v(t_{\min}) G_R(t, t_{\min}) + \int_{t_{\min}}^{t_{\max}} G(t, t_o) \left[\frac{d}{dt_o} + \eta\right] v(t_o) \tag{1.29}$$

The first term vanishes because $G_R(t, t_o)$ vanishes whenever $t < t_o$. The last term uses the equation of motion, Eq. (1.26), and the again the causality condition $G_R(t, t_o)$, leading to

$$v(t) = G_R(t, t_{\min}) v(t_{\min}) + \int_{t_{\min}}^{t} dt_o G_R(t, t_o) F(t_o) \tag{1.30}$$

In the absence of the external force we get

$$v(t) = G_R(t, t_{\min}) v(t_{\min}) \tag{1.31}$$

Self test:

(a) For a damped simple harmonic oscillator

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_o^2\right] x(t) = 0 \tag{1.32}$$

that the solution to the equation of motion (analogous to Eq. (1.31)) is

$$x(t) = m \left[G_R(t, t_o) \partial_{t_o} x(t_o) - x(t_o) \partial_{t_o} G_R(t, t_o)\right] + m\eta G_R(t, t_o) x(t_o) \tag{1.33}$$

(b) Check that Eq. (1.33) satisfies the equations of motion and the initial conditions. You will need to establish (by looking at the action of the adjoint) that

$$\lim_{t \to t_o} \left[m \partial_t \partial_{t_o} G_R(t, t_o) - m\eta \partial_t G_R(t, t_o)\right] = 0 \tag{1.34}$$

What are the modifications if the parameters m and η depend on time?