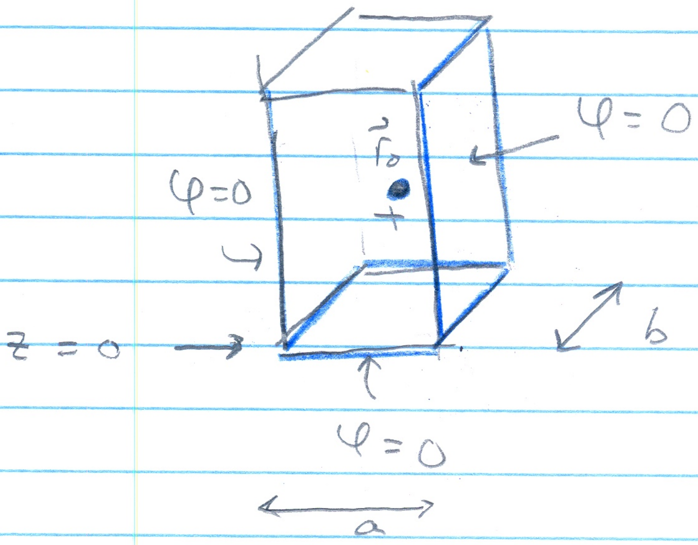
Solving For Green Fcns By Separation

Now consider a related problem



$G_D(\vec{r}, \vec{r}_0)$ = potential at $\vec{r}$ due to charge at $\vec{r}_0$

= find it by separation

- Consider a unit charge at $\vec{r}_0 = (\vec{x}_0, z_0)$ where $\vec{x}_0 = (x_0, y_0)$ denotes the coordinates in the plane.
- The walls of the box are grounded
- We wrote down the eigen-functions in the last problem

$$\psi_{nm}(\vec{x}) = \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$\int_0^a dx \int_0^b dy \; \psi^*_{nm}(\vec{x}) \, \psi_{n'm'}(\vec{x}) = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) \delta_{nn'} \delta_{mm'}$$

• They are complete

$$\bigstar \qquad \frac{4}{ab}\sum_{nm} \psi_{nm}(\vec{x})\,\psi^*_{nm}(\vec{x}_o) = \delta^2(\vec{x}-\vec{x}_o)$$

$$= \delta(x-x_o)\,\delta(y-y_o)$$

• Motivated by Eq $\bigstar$ we look a green-fcn of the form

$$G_D = \frac{4}{ab}\sum_{nm} g_{nm}(z,z_o)\,\psi_{nm}(\vec{x})\,\psi_{nm}(\vec{x}_o) \qquad (\bigstar\bigstar)$$

And will solve for $g(z,z_o)$.

• We substitute

$$-\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) G_D(\vec{r},\vec{r}_o) = \delta(z-z_o)\,\delta^2(\vec{x}-\vec{x}_o)$$

Since

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi_{nm}(\vec{x}) = \gamma^2_{nm}\,\psi_{nm}(\vec{x})$$

We find

$$\frac{4}{ab}\sum_{nm} \psi_{nm}(\vec{x})\psi_{nm}(\vec{x}_o)\left[-g''_{nm} + \gamma_{nm} g_{nm}\right]$$

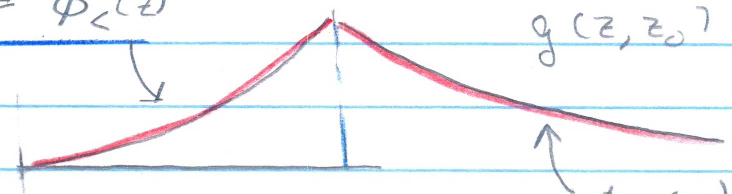
$$= \delta(z-z_o)\,\frac{4}{ab}\sum_{nm}\psi_{nm}(\vec{x})\psi_{nm}(\vec{x}_o)$$

Implying g_{nm} is a 1D Green fcn

Solving equations like this is discussed in a separate note

$$\bigstar \qquad \underbrace{\left[-\frac{\partial^2}{\partial z^2} + \gamma_{nm}^2 \right] g_{nm}(z,z_0) = \delta(z - z_0)}_{\text{1D Grn fcn equation}}$$

Then we may solve this equation as described on the couse web site



- Briefly for $z < z_0$ we solve ($\bigstar$)

$$g = A e^{-\gamma z} + B e^{\gamma z}$$

$$= A \underbrace{\sinh \gamma z}$$

$$\equiv \phi_<(z)$$ ← the solution for $z < z_0$

the potential vanishes at $z = 0$ (see problem statement)

- For $z > z_0$ we solve $\bigstar$

$$g = \underline{A} e^{-\gamma z} + \underline{B} \cancel{e^{\gamma z}} \quad B \to 0$$

we dont want growing solutions

$$= \underline{A} \underbrace{e^{-\gamma z}}$$

$$\equiv \phi_>(z)$$ ← the solution for $z > z_0$

- Continuity implies:

☆☆ $g = C\left[\phi_>(z)\,\phi_<(z_0)\,\Theta(z-z_0) + \phi_>(z_0)\,\phi_<(z)\,\Theta(z_0-z)\right]$

- Finally we integrate across the δ-fcn (see notes online)

☆ $$-\frac{\partial g}{\partial z}\bigg|_{z=z_0+\varepsilon} + \frac{\partial g}{\partial z}\bigg|_{z=z_0-\varepsilon} = 1$$

Which follows from

$$\int_{z_0-\varepsilon}^{z_0+\varepsilon}\left(-\frac{\partial^2}{\partial z^2}+\gamma^2\right)g = \int_{z_0-\varepsilon}^{z_0+\varepsilon}\delta(z-z_0)\,dz$$

- Substituting ☆☆ into ☆ we find

$$C\left[-\phi_>'(z_0)\,\phi_<(z_0) + \phi_<'(z_0)\,\phi_>(z_0)\right] = 1$$

or $C = \dfrac{1}{W(z_0)}$ whith $W(z_0) = \phi_<'\phi_> - \phi_<\phi_>'$

= wronskian of the two solutions

For $\phi_< = \sinh\gamma z$ and $\phi_> = e^{\gamma z}$,

$W = \gamma$.

Thus

$$g(z, z_0) = \frac{1}{\gamma}\left[\sinh\gamma z\, e^{-\gamma z_0}\,\Theta(z_0 - z) + \sinh\gamma z_0\, e^{-\gamma z}\,\Theta(z - z_0)\right]$$

- We will use a brief but obscure notation.

Let

$$z_< = \text{smaller of } z, z_0$$

$$= \begin{cases} z & z < z_0 \\ z_0 & z_0 < z \end{cases}$$

$$z_> = \text{greater of } z, z_0$$

$$= \begin{cases} z_0 & z < z_0 \\ z & z_0 < z \end{cases}$$

Same Thing!

Then

$$g(z; z_0) = \frac{1}{\gamma}\sinh\gamma z_<\, e^{-\gamma z_>}$$

Then

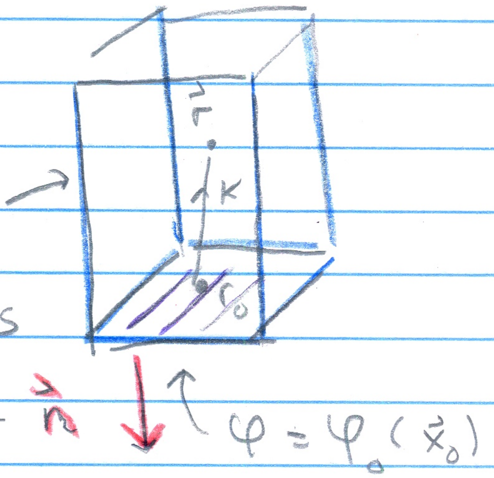
$$G_D(\vec{r}, \vec{r}_0) = \frac{4}{ab}\sum_{nm}\frac{1}{\gamma_{nm}}\sinh\gamma_{nm} z_<\, e^{-\gamma_{nm} z_>}\,\psi_{nm}(\vec{x})\,\psi^*_{nm}(\vec{x}_0)$$

First Comment

Solving the boundary problem with $G_D(\vec{r},\vec{r}_0)$

We will use G_D to construct the boundary propagator K

Returning to the first problem:
(see online!)



$\varphi = 0$

on sides

normal $= \vec{n}$

$$\varphi(\vec{r}) = \int dx_0\, dy_0\, K(\vec{r}, \vec{x}_0)\, \varphi_0(\vec{x}_0)$$

2D vector (x,y)

$$\vec{r} = (\vec{x}, z)$$

z coordinate

- We can construct the surface green-fcn K which will propagate the boundary data $\varphi_0(x,y)$ into the interior

$$K(\vec{r},\vec{r}_0) = -\vec{n}\cdot\nabla_{r_0} G_D(\vec{r},\vec{r}_0)\Big|_{\vec{r}_0\in\partial V} \qquad (\vec{n} = -\hat{z})$$

$$= \frac{\partial}{\partial z_0} G_D(\vec{r},\vec{r}_0)\Big|_{z_0=0}$$

- Using the Green fcn with $z_> = z$ and $z_< = z_0$ we have

$$\frac{\partial}{\partial z_0}\frac{1}{\gamma}\sinh\gamma z_0\Big|_{z_0=0} = 1 \quad \text{and} \quad \text{thus}$$

$$K = \frac{4}{ab} \sum_{nm} e^{-\gamma_{nm} z} \psi_{nm}(\vec{x}) \psi^*_{nm}(\vec{x}_0)$$

• Then according to the green-theorem

$$\varphi(\vec{r}) = \int dx_0 \, dy_0 \; K(\vec{r}, \vec{x}_0) \, \varphi_0(x_0, y_0)$$

Or

$$\varphi(\vec{r}) = \sum_{nm} e^{-\gamma_{nm} z} \psi_{nm}(\vec{x}) A_{nm}$$

Where

$$A_{nm} = \frac{4}{ab} \int dx_0 \, dy_0 \; \psi^*_{nm}(\vec{x}_0) \, \varphi_0(\vec{x}_0)$$

which agrees @ before

Second Comment: (see online!)

- The procedure we used to find the 1D green fcn generalizes. For the 1D equation

kind of measure

$$\star \quad \left[-\frac{d}{dx}\, p(x)\, \frac{d}{dx} + q(x)\right] g(x, x_0) = \delta(x - x_0)$$

- We can follow the same steps (see online). If $\phi_<(x)$ is the solution for $x < x_0$ and $\phi_>(x)$ is the solution for $x > x_0$. The Green function is

$$\star\star \quad g(x, x_0) = \frac{\left[\phi_<(x)\,\phi_>(x_0)\,\Theta(x_0 - x) + \phi_>(x)\,\phi_<(x_0)\,\Theta(x - x_0)\right]}{p(x_0)\, W(x_0)}$$

Where

$$W(x) = \phi_<'(x)\,\phi_>(x) - \phi_>'(x)\,\phi_<(x)$$

is the Wronskian. Also recall that

$$p(x)\, W(x) = \text{constant} \Rightarrow$$

So the denominator is actually a fixed number. For this example it was $1/\gamma$

so the denominator is not a functions of x

this is easy to prove from Eq $\star$ without the δ-fcn.

We will use a highly compact notation

$$g(x, x_0) = C\, \phi_<(x_<)\, \phi_>(x_>)$$

where

$$C = \frac{1}{pW}$$

Here $x_>$ is the greater of x and x_0, while $x_<$ is the lesser of x and x_0. This compact but cryptic notation means the same as the previous page, which should be used if confusion arises.