Problem 1. Lienard-Wiechert for constant velocity

(a) For a particle moving with constant velocity v along the $x-$axis show using Lorentz transformation that gauge potential from a point particle is

$$A^x(t, x, \boldsymbol{x}_\perp = \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + \gamma^2(x - vt)^2}} \tag{1}$$

at the point $(t, \boldsymbol{r}) = (t, x, y, z) = (t, x, \boldsymbol{b})$. So at the point $(t, 0, b, 0)$ the gauge potential A^x is is

$$A^x(t, x, y = b) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + (\gamma vt)^2}} \tag{2}$$

(b) Start by noting the definitions

$$T \equiv t - \frac{R}{c} \qquad R = |\boldsymbol{r} - \boldsymbol{r}_*(T)| \qquad \boldsymbol{R} \equiv R\boldsymbol{n} \equiv \boldsymbol{r} - \boldsymbol{r}_*(T) \qquad \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R} \tag{3}$$

and drawing a picture for yourself. Then, after setting $c = 1$ and $v = \beta$ to simplify algebra, show that the Lienard Wiechert result,

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[\frac{\mathbf{v}/c}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} . \tag{4}$$

gives the same result as Eq. (2).

(c) Show that the Lienard-Wiechert potential, Eq. (4), and analogous equation for φ can be written covariantly

$$A^\mu(X) = -\frac{e}{4\pi} \left[\frac{U^\mu}{U \cdot \Delta X} \right]_{\rm ret} , \tag{5}$$

where ΔX^μ is the difference in the space-time coordinate four vectors of the emission and observation points, and U^μ is the four velocity of the particle. What is $\Delta X \cdot \Delta X \equiv \Delta X^\mu \Delta X_\mu$? Can $[]_{\rm ret}$ be expressed covariantly?

Problem 2. Radiation during perpendicular acceleration

Consider an ultrarelativistic particle of velocity β experiencing an acceleration $a_\perp$ perpendicular to the direction of motion. Here $a_\perp$ points along the x-axis and $\boldsymbol{\beta}$ points along the z-axis.

(a) Show that the energy radiated per retarded time is approximately

$$\frac{dW}{dTd\Omega} = \frac{e^2}{16\pi^2 c^3}\frac{a_\perp^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \tag{6}$$

$$\simeq \frac{e^2}{2\pi^2 c^3}\frac{a_\perp^2}{(1+(\gamma\theta)^2)^3}\left[1-\frac{4(\gamma\theta)^2\cos^2\phi}{(1+(\gamma\theta)^2)}\right] \tag{7}$$

In the first equality, I give the full answer without approximation, but I will only grade the second approximate result.

Hint, in working out this radiation pattern you might (as a start) show without approximation that

$$|\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}|^2 = (1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2 a^2 - (\boldsymbol{n}\cdot\boldsymbol{a})^2(1-\beta^2) \tag{8}$$

by using the "b(ac)-(ab)c" rule. Then select a coordinate system were

$$\boldsymbol{\beta} = (0,0,\beta) \tag{9}$$

$$\boldsymbol{a} = (a_\perp, 0, 0) \tag{10}$$

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{11}$$

(b) Work in a ultra-relativistic approximation, and compute the total power by integrating over the solid angle (as done in class) to show that you obtain the appropriate relativistic Larmour result

$$\frac{dW}{dT} = \text{come on \ldots you know it \ldots right?} \tag{12}$$

Problem 3. An oscillator radiating

(a) Determine the time averaged power radiated per unit sold angle for a *non-relativistic charge* moving along the z-axis with instantaneous position, $z(T) = H\cos(\omega_o T)$.

(b) Now consider relativistic charge executing simple harmonic motion. Show that the instantaneous power radiated per unit solid angle is

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\,d\Omega} = \frac{e^2}{16\pi^2}\frac{c\beta^4}{H^2}\frac{\sin^2\theta\cos^2(\omega_o T)}{(1+\beta\cos\Theta\sin\omega_o T)^5} \tag{13}$$

Here $\beta = \omega_o H/c$ and $\gamma = 1/\sqrt{1-\beta^2}$

(c) In the relativistic limit the power radiated is dominated by the energy radiated during a short time interval around $\omega_o T = \pi/2,\, 3\pi/2,\, 5\pi/2,\, \ldots$. Explain why. Where does the outgoing radiation point at these times.

(d) Let ΔT denote the time deviation from one of this discrete times, *e.g.* $T = 3\pi/(2\omega_o) + \Delta T$. Show that close to one of these time moments:

$$\frac{dP(\Delta T)}{d\Omega} = \frac{dW}{d\Delta T\,d\Omega} \simeq \frac{2e^2}{\pi^2}\frac{c\beta^4}{H^2}\gamma^6\frac{(\gamma\omega_o\Delta T)^2(\gamma\theta)^2}{(1+(\gamma\theta)^2+(\gamma\omega_o\Delta T)^2)^5} \tag{14}$$

(e) By integrating the results of the previous part over the ΔT of a single pulse, show that the time averaged power is

$$\overline{\frac{dP(T)}{d\Omega}} = \frac{e^2}{128\pi^2}\frac{c\beta^4}{H^2}\gamma^5\frac{5(\gamma\theta)^2}{(1+(\gamma\theta)^2)^{7/2}} \tag{15}$$

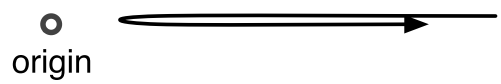
(f) Make rough sketches of the angular distribution for non-relativistic and relativistic motion.

Problem 4. Radiation during a collision

A classical non-relativistic charged particle of charge q and mass m is incident upon a repulsive mechanical potential $U(r)$

$$U(r) = \frac{\mathcal{A}}{r^2},$$

so that the force on the particle is $\boldsymbol{F} = -\nabla U(r)$. The particle moves along the x-axis and strikes the central potential head on as shown below. The incident kinetic energy (i.e. the kinetic energy of the particle far from the origin) is K.

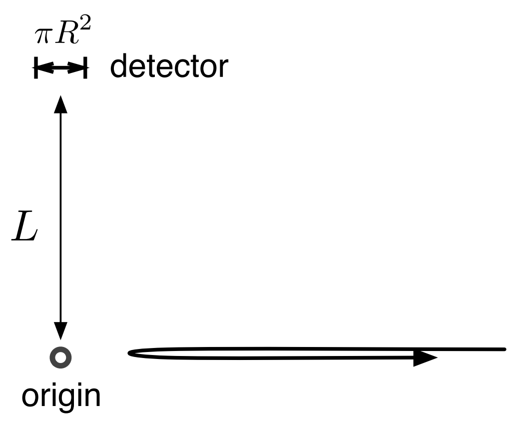


(a) (2 points) Determine the particle's classical trajectory $x(t)$. Adjust the integration constants so that the particle reaches its distance of closest approach at $t = 0$. Check that for late times $x(t)$ approaches $v_o\, t$ with the phyically correct value of v_o. Check that for small times $x(t)$ behaves as $x(t) \simeq x_o + \frac{1}{2}a_o t^2$ with the physically correct value of x_o.

(b) (4 points) Use dimensional reasoning and the Larmour formula to estimate the total energy lost to electromagnetic radiation during the collision. How does the energy lost scale with the incident velocity?

(c) (2 points) Calculate the energy lost to radiation during the collision processes. Some relevant integrals are given at the end of this problem.

Now consider a detector placed along the y-axis far from the origin as shown below. The front face of the detector has an area of πR^2, and the detector is placed at a distance L from the origin with $L \gg R$.



(d) (2 points) What is the direction of polarization of the observed light in the detector? Explain.

(e) (2 points) What is the typical frequency of the photons that are emitted at 90^o? Explain.

(f) (5 points) For the detector described above, determine the average number of photons received by the detector per unit frequency:

$$\frac{dN}{d\omega}\,. \tag{16}$$

Some relevant integrals are given at the end of the problem.

(g) (3 points) We have determined the photon radiation spectrum using classical electrodynamics. For what values of the parameters $\mathcal{A}$ and K is this approximation justified?

Useful integrals and formulas:

(a) For positive integer n, we note the integrals

$$\int_{-\infty}^{\infty} du \frac{1}{(1+u^2)^n} = \pi\, c_n \tag{17}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \ldots \tag{18}$$

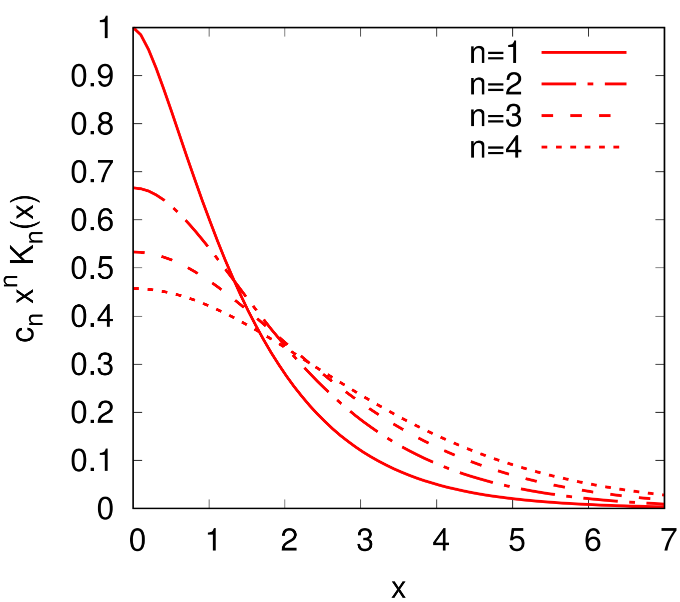
(b) For positive integers n, we note the integrals

$$\int_{0}^{\infty} du \frac{\cos(xu)}{(u^2+1)^{n+\frac{1}{2}}} = c_n\, x^n K_n(x) \tag{19}$$

where

$$c_1, c_2, c_3, c_4, \ldots = 1, \frac{1}{3}, \frac{1}{15}, \frac{1}{105}, \ldots \tag{20}$$

and $K_n(x)$ are the modified Bessel functions, and the RHS of Eq. (19) is illustrated below



Problem 5. Periodic pulses

Consider a periodic motion that repeats itself with period $\mathcal{T}_o$. Show that the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental, $\omega_o = 2\pi/\mathcal{T}_o$.

Let the electric field from a single pulse (or period) be $E_1(t)$, *i.e.* where $E_1(t)$ is non-zero between 0 and $\mathcal{T}_o$ and vanishes elsewhere, $t < 0$ and $t > \mathcal{T}_o$. Let $E_1(\omega)$ be its fourier transform.

(a) Suppose that the wave form repeats once so that two pulses are received. $E_2(t)$ consists of the first pulse $E_1(t)$, plus a second pulse, $E_2(t) = E_1(t) + E_1(t - \mathcal{T}_o)$. Show that the Fourier transform and the power spectrum is

$$E_2(\omega) = E_1(\omega)\,(1 + e^{i\omega\mathcal{T}_o}) \qquad |E_2(\omega)|^2 = |E_1(\omega)|^2\,(2 + 2\cos(\omega\mathcal{T}_o)) \tag{21}$$

(b) Now suppose that we have n (with n odd) arranged almost symmetrically around $t = 0$, *i.e.*

$$E_n(t) = E_1(t+(n-1)\mathcal{T}_o/2)+\ldots+E_1(t+\mathcal{T}_o)+E_1(t)+E_1(t-\mathcal{T}_o)+\ldots E_1(t-(n-1)\mathcal{T}_o/2)\,, \tag{22}$$

so that for $n = 3$

$$E_3(t) = E_1(t + \mathcal{T}_o) + E_1(t) + E_1(t - \mathcal{T}_o)\,. \tag{23}$$

Show that

$$E_n(\omega) = E_1(\omega)\,\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)} \tag{24}$$

and

$$|E_n(\omega)|^2 = |E_1(\omega)|^2\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right)^2 \tag{25}$$

(c) By taking limits of your expressions in the previous part show that after n pulses, with $n \to \infty$, we find

$$E_n(\omega) = \sum_m E_1(\omega_m)\frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m) \tag{26}$$

and

$$|E_n(\omega)|^2 = \underbrace{n\mathcal{T}_o}_{\text{total time}} \times \sum_m |E_1(\omega_m)|^2\frac{2\pi}{\mathcal{T}_o^2}\delta(\omega - \omega_m) \tag{27}$$

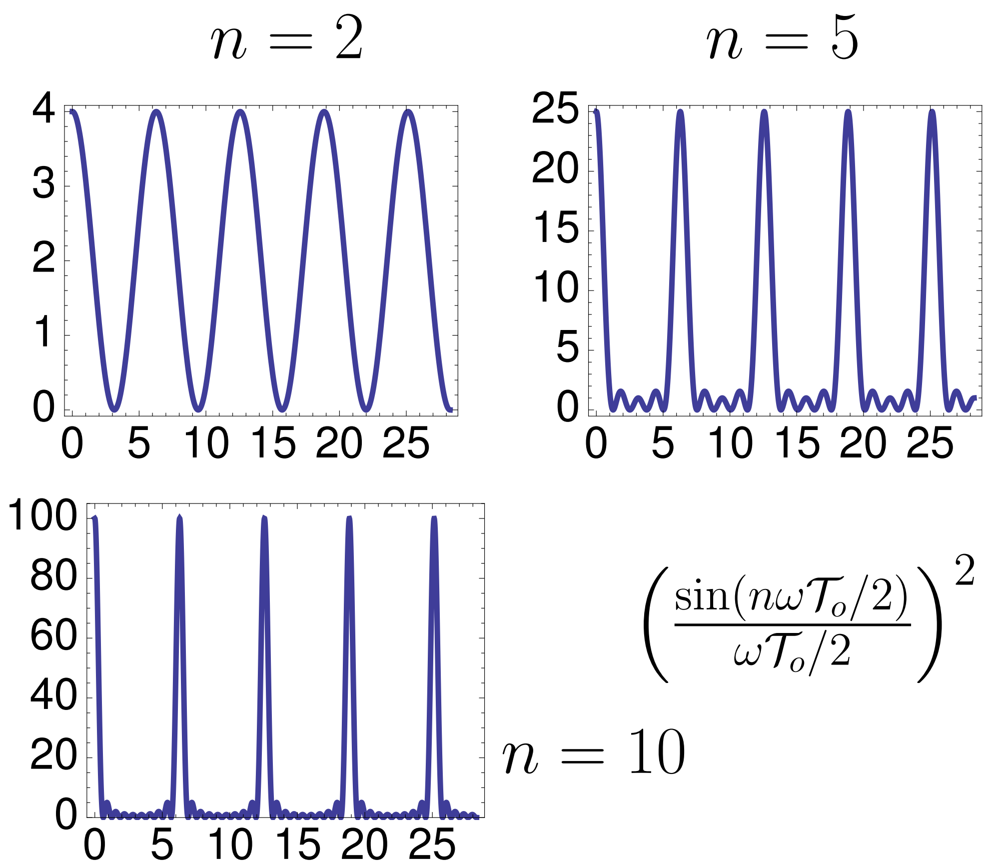
where $\omega_m = 2\pi m/\mathcal{T}_o$.

Remark We have in effect shown that if we define

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o)\,. \tag{28}$$

Then the Fourier transform of $\Delta(t)$ is

$$\hat{\Delta}(\omega) = \sum_n e^{-i\omega n\mathcal{T}_o} = \sum_m \frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m)\,. \tag{29}$$


$n = 2$
$n = 5$
$n = 10$
0 1 2 3 4
0 5 10 15 20 25
0 5 10 15 20 25
0 20 40 60 80 100
$\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\omega\mathcal{T}_o/2}\right)^2$

(d) Show that a general expression for the time averaged power radiated per unit solid angle into each multipole $\omega_m \equiv m\omega_o$ is:

$$\frac{dP_m}{d\Omega} = \frac{|rE(\omega_m)|^2}{\mathcal{T}_o^2} \tag{30}$$

Or

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2\omega_o^4 m^2}{32\pi^4 c^3} \left| \int_0^{\mathcal{T}_o} \mathbf{v}(T) \times \boldsymbol{n} \exp\left[i\omega_m (T - \frac{\boldsymbol{n} \cdot r_*(T)}{c}) \right] \right|^2 \mathrm{d}T \,, \tag{31}$$

Here $d\hat{P}_m/d\Omega$ is defined so that over along time period $\Delta\mathcal{T}$, the energy per solid angle is

$$\frac{dW}{d\Omega} = \Delta\mathcal{T} \sum_{m=1}^{\infty} \frac{d\hat{P}_m}{d\Omega} \tag{32}$$

Also note that we are summing only over the positive values of m which is different from how we had it in class:

$$\frac{d\hat{P}_m}{d\Omega} \equiv \frac{dP_m}{d\Omega} + \frac{dP_{-m}}{d\Omega} \tag{33}$$

Problem 6. Radiation spectrum of a SHO

(a) Show that for the simple harmonic motion of a charge discussed in Problem: An Oscillator Radiating, the average power radiated per unit solid angle in the m-th harmonic is

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2 c\beta^2}{8\pi^2 H^2} m^2 \tan^2\theta \left[J_m(m\beta\cos\theta)\right]^2 \tag{34}$$

(b) Show that in the non-relativistic limit the total power radiated is all in the fundamental and has the value

$$P = \frac{e^2}{4\pi}\frac{2}{3}\omega_o^4 \overline{H^2} \tag{35}$$

where $\overline{H^2}$ is the mean squared amplitude of the oscillation.

Problem 7. (Optional) Energy during a burst of deceleration

A particle of charge e moves at constant velocity, βc, for $t < 0$. During the short time interval, $0 < t < \Delta t$ its velocity remains in the same direction but its speed decreases linearly in time to zero. For $t > \Delta t$, the particle remains at rest.

(a) Show that the radiant energy emitted per unit solid angle is

$$\frac{dW}{d\Omega} = \frac{e^2\beta^2}{64\pi^2 c\Delta t} \frac{(2 - \beta\cos\theta)\left[1 + (1 - \beta\cos\theta)^2\right]\sin^2\theta}{(1 - \beta\cos\theta)^4} \tag{36}$$

(b) In the limit $\gamma \gg 1$, show that the angular distribution can be expressed as

$$\frac{dW}{d\xi} \simeq \frac{e^2\beta^2}{4\pi\, c} \frac{\gamma^4}{\Delta t} \frac{\xi}{(1+\xi)^4} \tag{37}$$

where $\xi = (\gamma\theta)^2$.

(c) Show for $\gamma \gg 1$ that the total energy radiated is in agreement with the relativistic generalization of the Larmour formula.