Problem 1. Soft bremsstrahlung during a decay

In a collision or decay that happens at location $\boldsymbol{r}_o$ over an infinitessimally short time scale, $\tau_{\rm accel}$, the charged particles moving with velocity, $\mathbf{v}_1, \mathbf{v}_2, \ldots$ before the collsions and the charged particles moving with $\mathbf{v}_{1'}, \mathbf{v}_{2'}, \ldots$, after the collision each contribute to the radiation field. (The total radiation field is just a sum of the radiation fields from each particle.)

(a) Show that for frequencies low $\omega \ll 1/\tau_{\rm accel}$ the total radiation field is

$$\boldsymbol{E}_{\rm rad}(\omega, r) = e^{i\omega(r-\boldsymbol{n}\cdot\boldsymbol{r}_o)/c}\left(\sum_{j'\,\in\,\text{final}} \frac{q_{j'}}{4\pi r c^2}\frac{\boldsymbol{n}\times\boldsymbol{n}\times\mathbf{v}_{j'}}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_{j'}} - \sum_{j\,\in\,\text{initial}} \frac{q_j}{4\pi r c^2}\frac{\boldsymbol{n}\times\boldsymbol{n}\times\mathbf{v}_j}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}_j}\right) \tag{1}$$

This generalizes the result of Lecture 46.

Hint. You may encounter an integral like

$$\int_0^\infty \boldsymbol{n}\times\boldsymbol{n}\times\mathbf{v}\; e^{i\omega T(1-\boldsymbol{n}\cdot\mathbf{v}/c)}\;. \tag{2}$$

To give this integral definite meaning insert a convergence factor $e^{-\epsilon|T|}$ and then take the limit $\epsilon \to 0$ after integration. In any real experiment the velocity $\mathbf{v}(T)$ would be cut off in time, and provide this convergence factor naturally.

(b) A neutral ω^o meson of mass $M_\omega c^2 = 784\,\text{MeV}$ has a relatively rare decay mode $\omega^o \to \pi^+\pi^-$, with branching fraction of 1.53%. (98.5% of the time it decays to something else.) It has another rare decay mode $\omega^o \to e^+e^-$ with branching ratio 7.28×10^{-3}%. (These are pretty rare decays for the ω^o meson – most of the time it decays to $\pi^+\pi^-\pi^0$ with a branching fraction of 89.2%). The mass of a pion is $mc^2 = 140\,\text{MeV}$, while the electron mass is ...

(i) Compute the frequency spectrum of the soft electromagnetic radiation per solid angle that accompanies both of these decay modes

$$\frac{dI}{d\omega d\Omega} = 2\left.\frac{dW}{d\omega d\Omega}\right|_{\omega>0}, \tag{3}$$

Describe your result qualitatively.

(ii) Show that for both of these decay modes the frequency spectrum of radiated energy at low frequencies is

$$\frac{dI}{d\omega} = \frac{e^2}{4\pi^2 c}\left[\left(\frac{1+\beta^2}{\beta}\right)\ln\frac{1+\beta}{1-\beta} - 2\right] \simeq \frac{e^2}{\pi^2 c}\left[\ln\left(\frac{M_\omega}{m}\right) - \frac{1}{2}\right] \tag{4}$$

where M_ω is the mass of the ω_o meson, m is the mass of one of the decay products, and β is the velocity/c of the decay products.

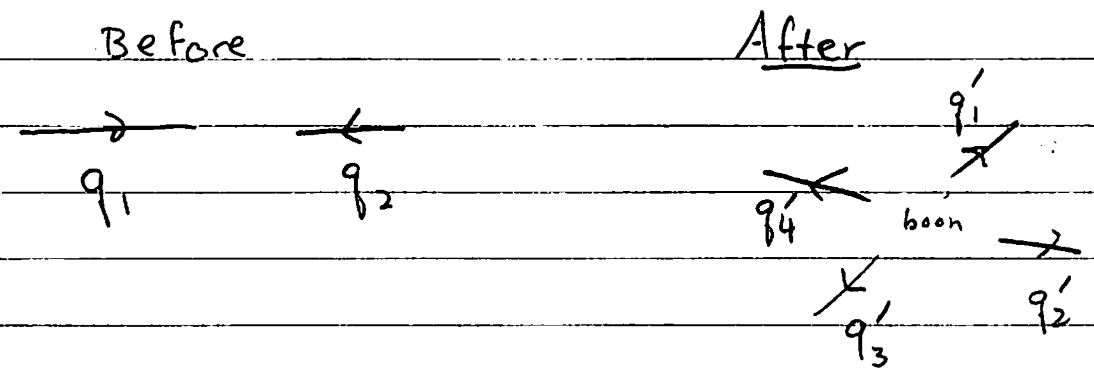
(iii) Roughly evaluate the total energy radiated in each decay by integrating the spectrum up to a point where the photon's momentum is half of the momentum of the decay products. (Beyond this point the recoil of the charged decay products would need to be considered. This lies outside of classical electrodynamics. In classical electrodynamics we specify the currents and solve for the fields.). You should find in a leading $\log(M_\omega/m)$ approximation

$$\frac{I_{\rm rough}}{M_\omega c^2} \simeq \frac{\alpha}{\pi} \log\left(\frac{M_\omega}{m}\right) \tag{5}$$

Using this rough evaluation, what fraction of the rest energy of the ω^o is carried away by soft radiation in the two decay modes

Bremsstrahlung

Consider a collision:



Then consider the radiation field from **final** particles. Take partice 1'

$$E_{rad}(\omega, r) = \frac{q_1'}{4\pi r c^2}\left(-i\omega e^{i\omega r/c}\right)\int_{-\infty}^{\infty} dT\, e^{i\omega(T - \vec{n}\cdot\vec{r}_*(T)/c)}\, n \times n \times v_1'(T)$$

The trajectory starts at $T=0$ and $r_*(T=0) = r_0$.
So the coordinates are:

$$r_{*1'}(T) = \vec{r}_0 + \vec{v}_{1'} T \quad \text{for } \underline{T > 0}$$

$$E_{rad,1'}(\omega, r) = \frac{q_{1'}}{4\pi r c^2}\left(-i\omega e^{i\omega r/c}\right)$$

$$\int_0^{\infty} dT\, e^{i\omega(T - \vec{n}\cdot r_0/c - \vec{v}_1\cdot\vec{n}/c\, T)}\, n \times n \times v_{1'}(T)$$

$$= \frac{q_{1'}}{4\pi r c^2}\, e^{i\frac{\omega}{c}(r - r_0\cdot n)}\,(-i\omega)\int_0^{\infty} e^{i\omega T(1 - \vec{n}\cdot\vec{v}/c)}\, n \times n \times \vec{v}_{1'}(T)$$

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Inserting the convergence factor and doing the integral:

$$I = -i\omega \int_0^{\infty} e^{i\omega T(1-\vec{n}\cdot\vec{v}_{1'}/c)}\, e^{-\varepsilon T}\; n\times n\times v_{1'}$$

$$= -i\omega \left. \frac{e^{-i\omega T(1-n\cdot v_{1'}/c)}\, e^{-\varepsilon T}}{(i\omega(1-n\cdot \frac{v}{c})-\varepsilon)} \right|_0^{\infty} n\times n\times v_{1'}$$

$$I = \frac{n\times n\times v_{1'}}{(1-n\cdot \frac{v}{c})}$$

So

$$E_{rad,1'}(\omega,r) = \frac{q_{1'}\, e^{i\frac{\omega}{c}(r-\vec{n}\cdot r_0)}}{4\pi r c^2}\; \frac{n\times n\times v_{1'}}{(1-\vec{n}\cdot\beta_{1'})}$$

Similarly from incoming particles the trajectory is

$$r_*(T) = r_0 + v_1 T \qquad \text{with} \qquad \underline{T<0}$$

So for the first particle

$$E_{rad,1}(r,\omega) = \frac{q}{4\pi r c^2} e^{i\omega/c\,(r-\vec{n}\cdot r_0)} \int_{-\infty}^{0} (-i\omega)\, e^{i\omega T(1-\vec{n}\cdot\vec{v}_1/c)}\; n\times n\times v_1(T)$$

Inserting a convergence factor $e^{+\varepsilon T}$ (since $T<0$ we must take $e^{+\varepsilon T}$) and doing the integral, which now contributes only at the upper limit $T=0$, Bremsstrahlung pg. 3

$$E_{rad,1}(r,\omega) = \frac{-q_1}{4\pi r c^2} e^{i\omega/c(\vec{r}-\vec{n}\cdot\vec{r}_0/c)} \frac{n \times n \times v_1}{(1 - n\cdot\frac{v}{c})}$$

So then for any real collision or decay

$$E_{rad}(r,\omega) = \sum_{i'} \frac{q_{i'}}{4\pi r c^2} e^{i\frac{\omega}{c}(r-\vec{n}\cdot\vec{r}_0)} \frac{n \times n \times v_{i'}}{(1 - n\cdot\beta_{i'})}$$

$$- \sum_{i} \frac{q_i}{4\pi r c^2} e^{i\frac{\omega}{c}(r-\vec{n}\cdot\vec{r}_0)} \frac{n \times n \times v_i}{(1 - n\cdot\beta_i)}$$

b) Using the result of part a)

$$\frac{dI}{d\omega\, d\Omega} = \frac{c\,|r E_{rad}(\omega, r)|^2}{\pi}$$

For the current problem the decay looks like this

before

neutral

ω^0

(no current)

after

π^- $\qquad$ π^+

$\frac{V_{2'}}{c} = -\beta\hat{z}$ $\qquad$ $\frac{V_{1'}}{c} = \beta\hat{z}$

So

$$\vec{V}_{1'} = c\beta\,\hat{z} \qquad q_{1'} = +Q$$

$$V_{2'} = -c\beta\,\hat{z} \qquad q_{2'} = -Q$$

Then

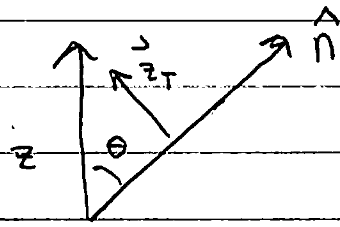
unimportant phase

$$E_{rad} = \frac{e^{i\phi}}{4\pi r c}\left[\frac{Q\; n\times n\times \beta\hat{z}}{(1-\beta\, n\cdot\hat{z})} + \frac{(-Q)\; n\times n\times(-\beta\hat{z})}{(1+\beta\, n\cdot\hat{z})}\right]$$

$$E_{rad} = \frac{e^{i\phi}}{4\pi r c}\left[\frac{2\beta Q\; n\times n\times\hat{z}}{(1-\beta^2(n\cdot\hat{z})^2)}\right]$$

norm of $-\vec{z}_T$

So taking $|\vec{n}\times\hat{n}\times\hat{z}| = \sin\Theta$



Then

$$\frac{c r^2 |E_{rad}(\omega, r)|^2}{\pi} = \boxed{\frac{Q^2}{16\pi^3 c}\left[\frac{4\beta^2\sin^2\Theta}{(1-\beta^2\cos^2\Theta)^2}\right] = \frac{dI}{d\omega\, d\Omega}}$$

The radiation spectrum is strongly peaked in the direction of the outgoing particles and is independent of frequency.

ii) To find the total spectrum we integrate over angle

$$\frac{dI}{d\omega} = \frac{Q^2}{4\pi^3 c}\int d\Omega\, \frac{\beta^2\sin^2\Theta}{(1-\beta^2\cos^2\Theta)^2}$$

$$= \frac{Q^2}{4\pi^3 c}\, 2\pi \int_{-1}^{1} dx\, \frac{\beta^2(1-x^2)}{(1-\beta^2 x^2)^2} \qquad x \equiv \cos\Theta$$

The integral is elementary, switching vars to $y = \beta x$

Bremsstrahlung pg. 6

$$\frac{dI}{d\omega} = \frac{Q^2}{2\pi^2 c}\left[\frac{1}{\beta}\int_{-\beta}^{\beta} dy \; \frac{-(1-\beta^2)}{(1-y^2)^2} + \frac{1}{(1-y^2)}\right]$$

$$= \frac{Q^2}{2\pi^2 c}\left[\frac{(1+\beta^2)}{2\beta}\log\frac{(1+\beta)}{(1-\beta)} - 1\right]$$

$$\boxed{\frac{dI}{d\omega} = \frac{Q^2}{4\pi^2 c}\left[\frac{(1+\beta^2)}{\beta}\log\frac{(1+\beta)}{(1-\beta)} - 2\right]}$$

$$\frac{dI}{d\omega} \simeq \frac{Q^2}{\pi^2 c}\left[\frac{2}{2}\log\sqrt{\frac{2}{1-\beta}} - \frac{1}{2}\right] \qquad \beta \to 1$$

using

$$1-\beta \simeq \frac{1}{2\gamma^2}$$

$$\frac{dI}{d\omega} \simeq \frac{Q^2}{\pi^2 c}\left[\log(2\gamma) - \frac{1}{2}\right]$$

$$E_{\omega^0} = E_{\pi^+} + E_{\pi^-}$$

$$m_{\omega^0} c^2 = 2E_\pi$$

$$\gamma_{\pi^+} = \frac{E_{\pi^+}}{m_\pi c^2} \simeq \frac{E_{\omega^0}/2}{mc^2} = \frac{1}{2}\frac{m_{\omega^0}}{m}$$

So

$$\frac{dI}{d\omega} \simeq \frac{Q^2}{\pi^2 c}\left[\log\left(\frac{m_{\omega^0}}{m}\right) - \frac{1}{2}\right]$$

iii) Then integrating over frequency $0 \ldots \omega_{max}$

$$I_{rough} \simeq \frac{Q^2}{\pi^2 c}\left[\log\left(\frac{m_{\omega^0}}{m}\right) - \frac{1}{2}\right]\omega_{max}$$

$$I_{rough} = \frac{Q^2}{4\pi\hbar c}(\hbar\omega_{max})\left[\frac{4}{\pi}\log\left(\frac{M_{\omega^0}}{m}\right) - \frac{2}{\pi}\right]$$

So taking, $\hbar\omega_{max} = \frac{E_\pi}{2} = \frac{M_\omega c^2}{4}$

The pions carry half the energy

Then with $\alpha = Q^2/4\pi\hbar c$

$$\frac{I_{rough}}{M_{\omega^0} c^2} = \alpha\frac{(M_\omega c^2/4)}{(M_\omega c^2)}\left[\frac{4}{\pi}\log\left(\frac{M_{\omega^0}}{m}\right) - \frac{2}{\pi}\right]$$

$$\frac{I_{rough}}{M_\omega c^2} = \alpha\left[\frac{1}{\pi}\log\left(\frac{M_{\omega^0}}{m}\right) - \frac{1}{2\pi}\right]$$

Taking a leading log approximation (we are already making a rough estimate)

$$\boxed{\frac{I_{rough}}{M_{\omega^0} c^2} \simeq \frac{\alpha}{\pi}\log\left(\frac{M_{\omega^0}}{m}\right)}$$

So plugging numbers

$$\log\left(\frac{M_{\omega^0}}{m_\pi}\right) = 1.72$$

$$\log\left(\frac{M_{\omega^0}}{m_e}\right) = 7.35$$

Evaluating:

$\omega \to \pi^+\pi^-$

$$\frac{I_{rough}}{M_{\omega^0}c^2} \simeq 0.4\%$$

$\omega \to e^+e^-$

$$\frac{I_{rough}}{M_{\omega^0}c^2} \simeq 1.7\%$$

Problem 2. Scattering from a perfectly conducting sphere

Consider light of wavenumber k scattering off a perfectly conducting sphere of radius a. Assume that $ka \ll 1$ and that the skin depth is much less than the size of the sphere The incident light propagates along the z-direction.

(a) **Optional** Show that the external field $\boldsymbol{E} = E_o e^{-i\omega t}\boldsymbol{\epsilon_o}$ and $\boldsymbol{H} = H_o e^{-i\omega t}\boldsymbol{n} \times \boldsymbol{\epsilon_o}$ induces a time dependent electric and magnetic dipole moment :

$$\boldsymbol{p} = 4\pi a^3 \boldsymbol{E}_o e^{-i\omega t} \qquad \boldsymbol{m} = -2\pi a^3 \boldsymbol{H}_o e^{-i\omega t} \tag{6}$$

For the magnetic case you can look at the solutions to homework 5 (pages 2-6). For the electric case you can look at lecture 3.

(b) By computing the radiated power from the time dependent magnetic and electric dipole, show that for arbitrary initial polarization ϵ_o of the incoming light, the scattering cross section off the sphere, summed over outgoing polarizations is given by:

$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_o, \boldsymbol{n}_o, \boldsymbol{n}) = k^4 a^6 \left[\frac{5}{4} - |\boldsymbol{\epsilon}_o \cdot \boldsymbol{n}|^2 - \frac{1}{4}|\boldsymbol{n} \cdot (\boldsymbol{n}_o \times \boldsymbol{\epsilon}_o)|^2 - \boldsymbol{n}_o \cdot \boldsymbol{n}\right] \tag{7}$$

where $\boldsymbol{n}_o$ and $\boldsymbol{n}$ are the directions of the incident and scattered radiations, while $\boldsymbol{\epsilon}_o$ is the (perhaps complex) unit polarization vector of the incident radiation ($\boldsymbol{\epsilon}_o^* \cdot \boldsymbol{\epsilon}_o = 1$; $\boldsymbol{n}_o \cdot \boldsymbol{\epsilon}_o = 0$).

Hint: as an intermediate step in the calculation show that

$$\boldsymbol{E}_{\text{rad}} = \frac{-\omega^2}{4\pi c^2} \frac{e^{-i\omega t + kr}}{r} D_o \left[-\epsilon_o + \boldsymbol{n}(\boldsymbol{n} \cdot \epsilon_o) - \tfrac{1}{2}\boldsymbol{n} \times (\boldsymbol{n}_o \times \epsilon_o)\right] \tag{8}$$

where $D_o = 4\pi a^3 E_o$. Then square this result (repeating to yourself like the the little engine ... "I think I can, I think I can, think I can") using the front cover of Jackson.

(c) If the incident radiation is linearly polarized, show that the cross section is

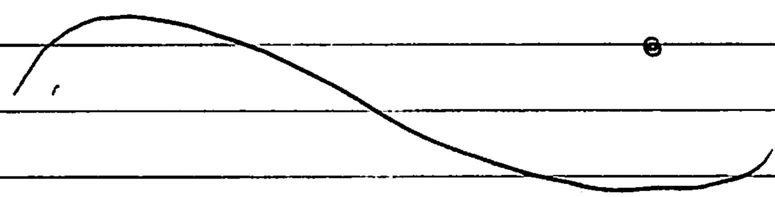
$$\frac{d\sigma}{d\Omega}(\boldsymbol{\epsilon}_o, \boldsymbol{n}_o, \boldsymbol{n}) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right] \tag{9}$$

where $\boldsymbol{n} \cdot \boldsymbol{n}_o = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

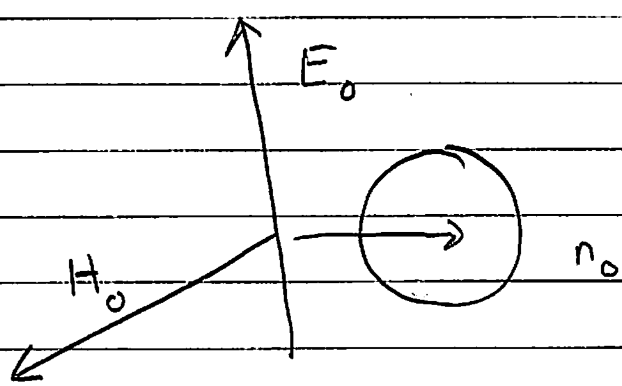
(d) What is the ratio of the scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

Problem - Scattering by a perfectly conducting sphere

Wave view



Sphere view



The time-dep electric field induces a time dependent electric dipole moment (Lecture 3)

$$\vec{p} = 4\pi a^3 E_0 e^{-i\omega t} \vec{\varepsilon}_0$$

The time dependent magnetic field induces eddy currents on the sphere surface which make a magnetic dipole moment (Homework 5 pgs. 2-6)

Then the induced magnetic moment

$$\vec{m} = -2\pi a^3 B_o \ \vec{n}_o \times \vec{\mathcal{E}}_o \ e^{-i\omega t}$$

note $B_o = E_o$

- This results follow fram a conducting sphere in a constant electric field (Lecture 3)
- And the induced in a spattially constant but time dependent magnetic field, ω

(Homework 5 pgs. 2-6)

Then we can calculate the radiation fields from these dipole moments

$$\vec{E}_{rad} = \frac{1}{4\pi r c^2}\left[n \times n \times \vec{p}(t_e) + \vec{n} \times \vec{m}(t_e) \right]$$

So with

$$\vec{p}\,(t_e) = p_o \, e^{-i\omega t + kr} \, \vec{\mathcal{E}}_o \qquad D_o \equiv \overbrace{4\pi a^3}^{\alpha_E} E_o$$

Then

$$\vec{E}_{rad} = \frac{1}{4\pi c^2} \frac{e^{-i\omega t + ikr}}{r} \left[D_o \, n \times n \times \mathcal{E}_o - \frac{D_o}{2} n \times n_o \times \mathcal{E}_o \right]$$

So

$$E_{rad} = \frac{-\omega^2}{4\pi c^2} \frac{e^{-i\omega t + ikr}}{r} D_0 \left[\underbrace{-\varepsilon_0}_{\equiv \vec{a}} + \underbrace{\vec{n}(n \cdot \varepsilon_0)}_{\equiv \vec{b}} \underbrace{- \frac{1}{2} n \times (n_0 \times \varepsilon_0)}_{\equiv \vec{c}} \right]$$

we want E^2_{rad}:

$$[\,]^2 = a^2 + b^2 + c^2 + 2a \cdot b + 2a \cdot c + 2\cancel{b} \cdot c$$

$$= 1 + (n \cdot \varepsilon_0)^2 + \frac{1}{4}(n \times n_0 \times \varepsilon_0)^2 - 2(\varepsilon_0 \cdot n)^2 + \varepsilon_0 \cdot (n \times (n_0 \times \varepsilon_0))$$

$$= 1 - (\varepsilon_0 \cdot n)^2 + \underbrace{\frac{1}{4}(n \times n_0 \times \varepsilon_0)^2}_{\mathrm{I}} + \underbrace{\varepsilon_0 \cdot (n \times (n_0 \times \varepsilon_0))}_{\mathrm{II}}$$

Tackling:

$$\mathrm{I} = \frac{1}{4}(n \times (n_0 \times \varepsilon_0)) \cdot (n \times (n_0 \times \varepsilon_0)) \qquad (a \times b) \cdot (c \times d)$$

$$= \frac{1}{4}(n_0 \times \varepsilon_0)^2 - \frac{1}{4}|n \cdot (n_0 \times \varepsilon_0)|^2 \qquad (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

$$\mathrm{I} = \frac{1}{4} - \frac{1}{4}|\vec{n} \cdot (\vec{n}_0 \times \vec{\varepsilon}_0)|^2$$

while:

$$\mathrm{II} = \varepsilon_0 \cdot (n \times (n_0 \times \varepsilon_0))$$

$$= \varepsilon_0 \cdot \left[(n \cdot \varepsilon_0)\vec{n}_0 - (n \cdot n_0)\vec{\varepsilon}_0 \right]$$

$$= -n \cdot n_0$$

So

$$[\;]^2 = 1 - (\varepsilon_0 \cdot n)^2 + \frac{1}{4} - \frac{1}{4}|n\cdot(n_0 \times \varepsilon_0)|^2 - n\cdot n_0$$

$$[\;]^2 = \frac{5}{4} - (\varepsilon_0 \cdot n)^2 - \frac{1}{4}|n\cdot(n_0 \times \varepsilon_0)|^2 - n\cdot n_0$$

Then

$$\frac{d\overline{P}}{d\Omega} = c\,\frac{(D_0^2/2)}{16\pi^2}\left(\frac{\omega}{c}\right)^4 \overline{[\;]}^2$$

So using

$$D_0 = 4\pi a^3 E_0$$

and

$$\frac{d\sigma}{d\Omega} = \frac{d\overline{P}/d\Omega}{\frac{c}{2}|E_0|^2}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\omega}{c}\right)^4 a^6 \left[\frac{5}{4} - (\varepsilon_0\cdot n)^2 - \frac{1}{4}|n\cdot(n_0\times\varepsilon_0)|^2 - n\cdot n_0\right]}$$

Then for linear polarization

$$n = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\vec{\varepsilon}_0 = (1, 0, 0)$$

$$\vec{n}_0 \times \vec{\varepsilon}_0 = (0, 1, 0)$$

So

$$\frac{d\sigma}{d\Omega} = \left(\frac{\omega}{c}\right)^4 a^6 \left[\frac{5}{4} - \sin^2\theta\,\overbrace{\cos^2\phi}^{\frac{1+\cos 2\phi}{2}} - \frac{1}{4}\sin^2\theta\,\overbrace{\sin^2\phi}^{\frac{1-\cos 2\phi}{2}} - \cos\theta\right]$$

Well ok, using $\cos 2\phi = 2\cos^2\phi - 1 = 1 - 2\sin^2\phi$

So

$$\frac{d\sigma}{d\Omega} = \left(\frac{\omega}{c}\right)^4 a^6 \left[\frac{5}{4} - \frac{5\sin^2\theta}{8} - \frac{3}{8}\sin^2\theta\cos 2\phi - \cos\theta\right]$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{\omega}{c}\right)^4 a^6 \left[\frac{5(1+\cos^2\theta)}{8} - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]}$$

d)

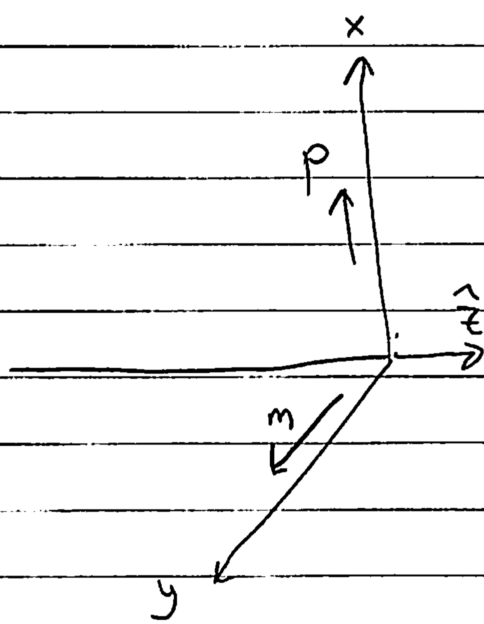
At $\theta = \pi/2 \quad \phi = 0$ (on x-axis)

$$\frac{d\sigma}{d\Omega} \rightarrow k^4 a^6 \overbrace{\frac{1}{4}}^{5/8 - 3/8}$$

While at $\theta = \pi/2 \quad \phi = \pi/2$ (on y-axis)

$$\frac{d\sigma}{d\Omega} \rightarrow k^4 a^6 \left[\frac{5}{8} + \frac{3}{8}\right] = k^4 a^6$$

The ratio is $\frac{1}{4}$ because the magnetic dipole radiation contributes to the radiation field along the x-axis (it is Larmour like), while on the y-axis only the electric dipole contributes to the radiation. The magnetic dipole moment is half the size of the electric dipole moment, and the square of the moment contributes to the power



Problem 3. Thomson Scattering (Optional. Done in class)

We will do this in class. It is very important, especially for astrophysics.

(a) Polarized light with linear polarization vector $\boldsymbol{\epsilon}_o$, is propagating in the z-direction with electric field amplitude E_o and is incident upon an electron at rest. Assume that $\hbar\omega$ is much less than the electron mass $m_e c^2$. Show that the time average power radiated into light with polarization $\boldsymbol{\epsilon}$ is

$$\left\langle \frac{dP_{\rm pol}}{d\Omega} \right\rangle = \tfrac{1}{2} c E_o^2 \left(\frac{e^2}{4\pi\, m_e c^2} \right)^2 |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_o|^2 \tag{10}$$

where $\boldsymbol{\epsilon}$ is the polarization of the outgoing radiation, *i.e.* $\boldsymbol{n} \cdot \boldsymbol{\epsilon} = \boldsymbol{z} \cdot \boldsymbol{\epsilon}_o = 0$.

(b) Show that the time averaged power radiated into light of any polarization by an incident beam with polarization ϵ_o is

$$\left\langle \frac{dP_{\rm unpol}}{d\Omega} \right\rangle = \tfrac{1}{2} c E_o^2 \left(\frac{e^2}{4\pi\, m_e c^2} \right)^2 |\boldsymbol{n} \times \boldsymbol{\epsilon}_o|^2 \tag{11}$$

(c) Show that the polarized and unpolarized cross sections for incident light with polarization $\boldsymbol{\epsilon}_o$ are

$$\frac{d\sigma_{\rm pol}}{d\Omega} = r_e^2 \, |\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_o|^2 \tag{12}$$

and

$$\frac{d\sigma_{\rm unpol}}{d\Omega} = r_e^2 \, |\boldsymbol{n} \times \boldsymbol{\epsilon}_o|^2 \ , \tag{13}$$

respectively. Here the classical electromagnetic radius is

$$r_e = \frac{e^2}{(4\pi) m_e c^2} \tag{14}$$

(d) By sticking in appropriate powers of $\hbar$, show that r_e is 137 times smaller than the compton wavelength, $\lambda\!\!\!^{-}_C = \hbar/m_e c$. Show that r_e is $(137)^2$ times smaller than the Bohr radius.

Remark: A heuristic way to understand why r_e is smaller than the "the size of an electron" , $\hbar/m_e c$, is that the cross section is the cross-sectional area $\propto (\hbar/m_e c)^2$ of the electron times the probability that the light will actually interact with the electron, wich is α^2.

(e) Now consider unpolarized incident light (light which is equally likely to be polarized in the x or y directions). Let the radiation be scattered at an angle θ in the xz plane, where $\boldsymbol{n} \cdot \boldsymbol{n}_o = \cos\theta$. Depending on the scattering angle θ, the outgoing light will be partially polarized in the xz plane, or out of the xz plane (*i.e.* in the y direction).

Show that the cross-section for unpolarized light to produce in-plane polarized light is

$$\frac{d\sigma_{\parallel}}{d\Omega} = \tfrac{1}{2} r_e^2 \cos^2\theta \tag{15}$$

while the cross-section to produce out-of-plane polarized light is

$$\frac{d\sigma_\perp}{d\Omega} = \tfrac{1}{2} r_e^2 \tag{16}$$

And conclude that the cross-section for unpolarized light to produce light of any polarization is

$$\frac{d\sigma}{d\Omega} = r_e^2 \frac{1 + \cos^2\theta}{2} \tag{17}$$

(f) By using the results of this problem and integrating over angles, or appealing directly to the Larmour formula, determine the total electromagnetic cross section for light electron scattering. This is known as the Thomson cross section:

$$\sigma_T = \frac{8\pi}{3} r_e^2 \tag{18}$$

Evaluate the Thomson cross section numerically, without looking up any numbers.

(g) Plot the polarization asymmetry

$$\frac{\frac{d\sigma_\parallel}{d\Omega} - \frac{d\sigma_\perp}{d\Omega}}{\frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega}} \tag{19}$$

as a function of scattering angle θ.

Thomson Scattering

For Thomson Scattering:

The incoming field drives acceleration of electron

$$\vec{E}_{inc} = E_0\, \varepsilon_0\, e^{ikz - i\omega t}$$

$$\vec{a} = \frac{F}{m} = \frac{q\vec{E}_{inc}}{m} = \frac{qE_0\vec{\varepsilon}_0}{m}\, e^{-i\omega t}$$ (we assume that the electron is at $z=0$)

Then

$$\vec{E}_{rad} = \frac{q}{4\pi r c^2}\, n\times n\times \vec{a}$$

$$\star \quad r\vec{E}_{rad} = \frac{q}{4\pi c^2}\, \frac{qE_0}{m}\, e^{-i\omega t}\, \vec{n}\times\vec{n}\times\vec{\varepsilon}_0$$

$$\star\star \quad \varepsilon^* \cdot r\vec{E}_{rad} = \left(\frac{q^2}{4\pi mc^2}\right) E_0\, e^{-i\omega t}\, \vec{\varepsilon}^* \cdot (\vec{n}\times\vec{n}\times\vec{\varepsilon}_0)$$

Then note: $\varepsilon^* \cdot (\vec{n}\times\vec{n}\times\varepsilon_0) = \vec{\varepsilon}^* \cdot \vec{\varepsilon}_0$ (see Lecture 49 pg. 3)

a) $$\boxed{\frac{dP(\varepsilon)}{d\Omega} = \frac{cE_0^2}{2}\left(\frac{q^2}{4\pi mc^2}\right)^2 |\varepsilon^* \cdot \varepsilon_0|^2 = \frac{c}{2}\, |\vec{\varepsilon}^* \cdot E_{rad}|^2 r^2}$$

Thomson Scattering pg. 2

b) So the unpolarized rate is

$$\overline{\frac{dP}{d\Omega}} = \frac{c}{2} |E_{rad}|^2 r^2$$

$$\frac{d\bar{P}}{d\Omega} = \frac{cE_0^2}{2}\left(\frac{q^2}{4\pi mc^2}\right)^2 |n\times n\times \varepsilon_0|^2$$

Now

$$|n\times n\times \varepsilon_0|^2 = |n\times\varepsilon_0|^2$$

So $$\boxed{\frac{d\bar{P}}{d\Omega} = \frac{cE_0^2}{2}\left(\frac{q^2}{4\pi mc^2}\right)^2 |\vec{n}\times\vec{\varepsilon}_0|^2}$$ (b)

c) The cross sections are just the radiated powers divided by the incoming flux

$$\overline{\vec{S}_{inc}\cdot\hat{z}} = \frac{c}{2}|E_0|^2$$

$$\frac{d\sigma}{d\Omega}_{pol} = r_e^2\, |\varepsilon^*\cdot\varepsilon_0|^2 \qquad r_e = \frac{q^2}{4\pi mc^2}$$

$$\frac{d\sigma}{d\Omega}_{unpol} = r_e^2\, |\vec{n}\times\vec{\varepsilon}_0|^2$$

Thomson Scattering pg. 3

Now

d)

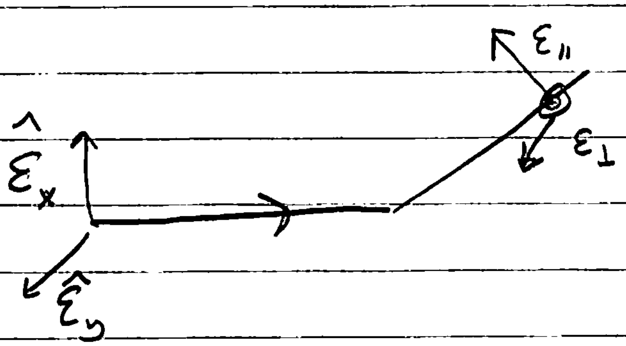
$$r_e = \left(\frac{q^2}{4\pi \hbar c}\right) \frac{\hbar}{mc}$$

$$r_e = \frac{1}{137} \frac{\hbar}{mc}$$

$$= \frac{1}{137} \alpha \left(\frac{\hbar}{mc\alpha}\right) = \frac{1}{(137)^2} \overbrace{\left(\frac{\hbar}{mc\alpha}\right)}^{\equiv a_0}$$

$$r_e = \frac{1}{(137)^2} a_0$$

e) $$\frac{d\sigma}{d\Omega}(\vec{\varepsilon}; \vec{\varepsilon}_0) = r_e^2 \, |\varepsilon^* \cdot \varepsilon_0|^2$$



Notation:

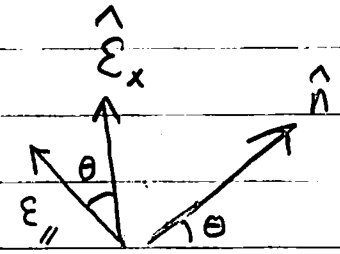
- $\hat{\varepsilon}_x$ and $\hat{\varepsilon}_y \equiv$ two incoming pol. vectors
- $\varepsilon_{\parallel} + \varepsilon_{\perp} \equiv$ two outgoing pol. vectors

$$\frac{d\sigma_{\parallel}}{d\Omega} = \underbrace{\frac{1}{2}\frac{d\sigma}{d\Omega}(\varepsilon_{\parallel}; \varepsilon_x)} + \underbrace{\frac{1}{2}\frac{d\sigma}{d\Omega}(\varepsilon_{\parallel}; \varepsilon_y)}$$

$$= \frac{1}{2} r_e^2 \, |\varepsilon_{\parallel} \cdot \varepsilon_x|^2 + 0$$

Thomson Scattering pg. 4

So we find



$$\boxed{\frac{d\sigma_{\parallel}}{d\Omega} = \frac{1}{2} r_e^2 \cos^2\theta}$$

And Similarly

$$\frac{d\sigma_{\perp}}{d\Omega} = \underbrace{\frac{1}{2}\frac{d\sigma}{d\Omega}(\varepsilon_{\perp}; \varepsilon_x)}_{0} + \underbrace{\frac{1}{2}\frac{d\sigma}{d\Omega}(\varepsilon_{\perp}; \varepsilon_y)}_{\frac{1}{2} r_e^2 |\varepsilon_{\perp}^* \cdot \varepsilon_y|^2}$$

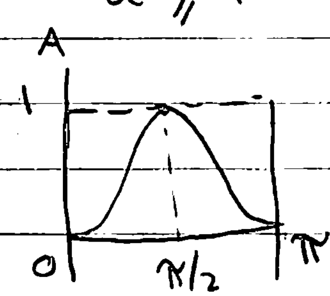
$$\boxed{\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2} r_e^2}$$

So

$$\boxed{\frac{d\sigma}{d\Omega} = r_e^2 \left(\frac{1+\cos^2\theta}{2} \right)} \leftarrow \text{Sum of two cross sections}$$

f)

$$A = \frac{d\sigma_{\parallel} - d\sigma_{\perp}}{d\sigma_{\parallel} + d\sigma_{\perp}} = \frac{(1-\cos^2\theta)}{(1+\cos^2\theta)}$$



Problem 4. (Optional) Radiation spectrum from a damped SHO

The non-relativistic motion of a charged particle of charge e is described by a damped harmonic oscillator

$$m\frac{d^2z}{dt^2} + m\eta\frac{dz}{dt} + m\omega_o^2 z = 0 \tag{20}$$

where η is small, $\eta \ll \omega_o$. Also assume that $\Delta\omega \equiv \omega - \omega_o \ll \omega_o$. Be sure to use these approximations at all points of the clculation.

The charge is released from rest with initial amplitude $z(t = 0) = H$.

(a) On the x axis, far from the charge, how is the light polarized ?

(b) Estimate (i.e. don't calculate) the energy lost per time to radiation. We will require that the energy lost to radation is small compared to energy lost to friction. How does this requirement constrain the dimensionful parameters of this problem: $m, H, \omega_o, \eta, e, c$

(c) Determine the spectrum of photons which are emitted

$$\omega\frac{dN}{d\omega} = \frac{1}{\hbar}\frac{dI}{d\omega} = \frac{2}{\hbar}\left.\frac{dW}{d\omega}\right|_{\omega>0} \tag{21}$$

(The factor of two incorporates the contributions with $\omega < 0$, which give an equal contribution. Why?) Express your final result in terms of the fine structure constant α instead of the charge (squared).

(d) **Optional – but extremely good practice for exam** Integrate the results of the previous part over frequency to determine the total energy that is emitted. Calculate the same result by integrating the Larmour formula

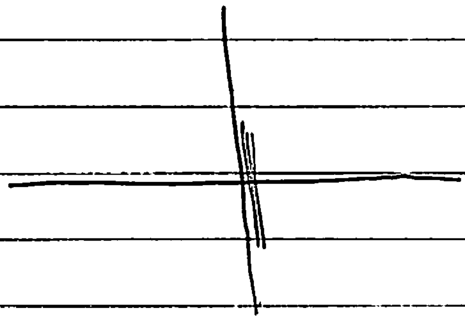
$$P(t_e) = \frac{q^2}{4\pi}\frac{2}{3}\frac{a^2(t_e)}{c^3} \tag{22}$$

over time.

(e) **Optional** In part (c) you determine the frequency spectrum for $\Delta\omega \ll \omega_o$. In part (d) you integrated over $\Delta\omega$ (from $-\infty \ldots \infty$) to determine the total power. Estimate the error made by extending this integral over the full frequency range instead of just a narrow range around ω_o. Similarly estimate the error in your approximate formula for the acceleration.

Radiation Spectrum from a Damped SHO

a)



$$E_{rad} = \frac{q}{4\pi r c^2} \vec{n} \times \vec{n} \times \vec{a}(t_e) = \frac{q}{4\pi r c^2}(-\vec{a}_T)$$

Since $\vec{n}$ is along the x-axis, while $\vec{a}$ is on the z axis, $\vec{a}$ is already transverse to $\vec{n}$. Thus $\vec{a}_T = \vec{a}$ in this case. The electric field is polarized along $\hat{z}$

b) $$P = \frac{e^2}{4\pi} \frac{2}{3} \frac{a^2}{c^3}$$

$$x = He^{-\frac{\eta}{2}t} \cos \omega_0 t$$

small compared to leading term $\propto \omega_0$

$$\dot{x} = -H\omega_0 e^{-\frac{\eta}{2}t} \sin \omega_0 t + O(\eta)$$

$$\ddot{x} = -H\omega_0^2 e^{-\frac{\eta}{2}t} \cos \omega_0 t + O(\eta)$$

Requiring the radiation be small

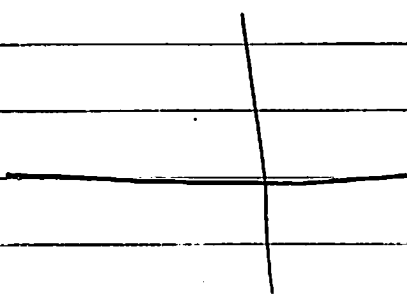
$$\frac{e^2}{4\pi}\frac{(H\omega_0^2)^2}{c^3} \ll \underbrace{\vec{F}_D \cdot \vec{v}}_{\text{energy lost to friction}}$$

$$\frac{e^2}{4\pi} H^2 \frac{\omega_0^4}{c^3} \ll m\eta(H\omega_0)(H\omega_0)$$

$$\frac{e^2}{4\pi m c^2}\left(\frac{\omega_0^2}{c}\right) \ll \eta$$

$$\boxed{\frac{r_e}{c}\omega_0^2 \ll \eta}$$ where $r_e = \dfrac{e^2}{4\pi m c^2}$

c) So



$$2\pi \frac{dW}{d\omega\, d\Omega} = c\,|r\,E_{rad}(\omega)|^2$$

So

$$E_{rad}(\omega, r) = \frac{q}{4\pi r c^2} \int_{-\infty}^{\infty} n \times n \times a(t_e)\, e^{i\omega t}\, dt$$

where $t_e = t - \frac{r}{c}$. We would rather integrate over t_e.

For instance, the partice starts accelerating at $t_e = 0$, while the light first arrives at the observation pnt at time $t = r/c$.
Switching vars to $t_e = t - r/c$

$$E_{rad}(\omega, r) = \frac{q}{4\pi r c^2} \int_{-\infty}^{\infty} n \times n \times \vec{a}(t_e)\, e^{i\omega(t_e + r/c)}\, dt_e$$

Or

$$E_{rad}(\omega, r) = \frac{q}{4\pi r c^2} e^{i\omega r/c} (\hat{n} \times \vec{n} \times \hat{z}) \underbrace{\int_0^{\infty} dt_e \left(-H\omega_0^2 e^{-\eta t/2} \cos\omega_0 t_e\right) e^{i\omega t_e}}_{I}$$

Handling the integral using
$\cos\omega_0 t_e = \frac{1}{2}(e^{i\omega t_e} + e^{-i\omega t_e})$ we find

$$I = \int_0^{\infty} -\frac{H\omega_0^2}{2} \left(e^{-\eta t_e/2} e^{i(\omega_0 + \omega) t_e} + e^{-\eta t_e/2} e^{i(\omega - \omega_0) t_e} \right)$$

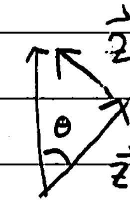
$$= -\frac{H\omega_0^2}{2} \left[\frac{1}{-(\eta/2) + i(\omega_0 + \omega)} + \frac{1}{-(\eta/2) + i(\omega - \omega_0)} \right]$$

For $\omega - \omega_0 \sim \eta$ we neglect

$$\frac{1}{\eta/2 + i(\omega_0+\omega)}$$

in comparison to

$$\frac{1}{\eta/2 + i(\omega-\omega_0)}$$



$\hat{z}$

$\vec{z}_T \Rightarrow |\vec{z}_T| = \sin\theta$

$\vec{z}_\parallel \Rightarrow |\vec{z}_\parallel| = \cos\theta$

Then

$$E_{rad} = \frac{q}{4\pi r c^2} e^{i\omega r/c}\, \overbrace{n\times n\times\hat{z}}^{\text{(transverse piece of } \hat{z})} \left(\frac{H\omega_0^2}{2}\right) \frac{1}{\left(-\frac{\eta}{2} + i(\omega_0-\omega)\right)}$$

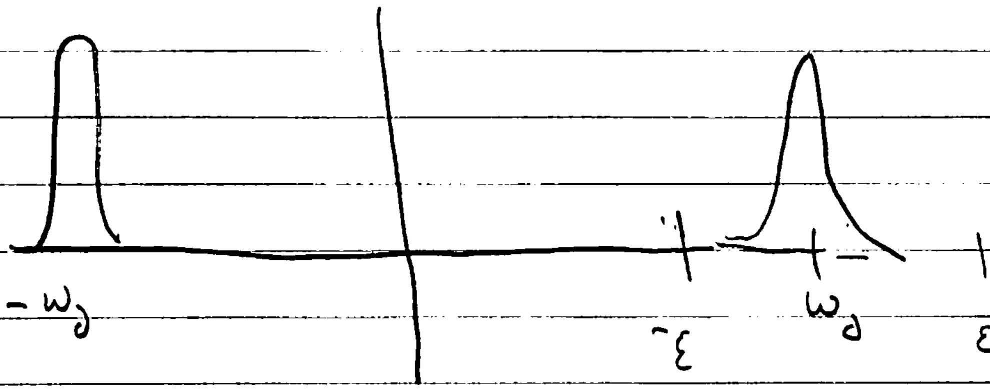
So

$$2\pi \frac{dW}{d\omega\, d\Omega} = c\,|E_{rad}(\omega)|^2$$

$$2\pi \frac{dW}{d\omega\, d\Omega} = \frac{q^2}{16\pi^2 c^3} \sin^2\theta \; (H\omega_0^2/2) \frac{1}{\left(\frac{\eta}{2}\right)^2 + (\omega_0-\omega)^2}$$

Then $$2\pi \frac{dW}{d\omega} = \frac{q^2}{16\pi^2 c^3}\left(\frac{8\pi}{3}\right) \frac{(H\omega_0^2/2)^2}{\left(\left(\frac{\eta}{2}\right)^2 + (\omega_0-\omega)^2\right)}$$

d) Plot $dW/d\omega$



Integrating around the peak from $-\varepsilon \dots \varepsilon$:

$$I = \int_{\omega_0-\varepsilon}^{\omega_0+\varepsilon} \frac{d\omega}{2\pi} \frac{1}{\left(\left(\frac{\eta}{2}\right)^2 + (\omega-\omega_0)^2\right)}$$

Then changing variables to $\left(\frac{\omega-\omega_0}{\eta/2}\right) \equiv x$

$$I = \frac{1}{(\eta/2)} \int_{-\varepsilon/(\eta/2)}^{+\varepsilon/(\eta/2)} \frac{dx}{2\pi} \frac{1}{(1+x^2)} \simeq \frac{1}{\eta/2} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{1}{(1+x^2)} = \frac{1}{\eta}$$

Since, $\frac{\eta}{2} \ll \varepsilon \ll \omega_0$, $\varepsilon/(\eta/2)$ is a large number. To leading order in η/ω_0 we can extend the upper limit to infinity

So the energy

$$W = 2 \int_0^\infty \frac{q^2}{16\pi^2 c^3} \left(\frac{8\pi}{3}\right) \frac{(H\omega_0^2/2)^2}{((\eta/2)^2 + (\omega_0 - \omega)^2)^2} \frac{d\omega}{2\pi}$$

accounts for the range $-\infty \ldots 0$

$$W = 2 \frac{q^2}{4\pi c^3}\left(\frac{2}{3}\right) (H\omega_0^2/2)^2 \cdot \frac{1}{\eta} = \frac{q^2}{4\pi c^3}\left(\frac{2}{3}\right) \frac{(H\omega_0^2)^2}{2\eta}$$

Now we do it another way

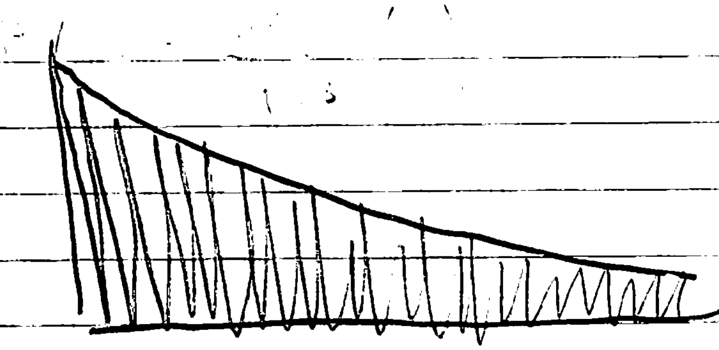
$$P(t_e) = \frac{q^2}{4\pi c^3} \cdot \frac{2}{3} a^2(t_e)$$

$$P(t_e) = \frac{q^2}{4\pi c^3} \frac{2}{3} \left(H\omega_0^2 e^{-\eta/2 t_e} \cos \omega_0 t_e\right)^2$$

Integrating over time from $t = 0 \ldots \infty$

$$W = \frac{q^2}{4\pi c^3}\left(\frac{2}{3}\right) (H\omega_0^2)^2 \underbrace{\int_0^\infty e^{-\eta t_e} \cos^2 \omega_0 t_e \, dt_e}_{\text{Integrand}}$$

Then plot the integrand



Since the $\cos^2 \omega_0 t_e$ oscillates many times over the decay time we can replace $\cos^2 \omega_0 t$ by its average value $= \frac{1}{2}$, to leading order in η / ω_0

$$\int_0^\infty e^{-\eta t} \cos^2 \omega_0 t \, dt \simeq \int_0^\infty e^{-\eta t} \frac{1}{2} \, dt$$

$$= \frac{1}{2\eta}$$

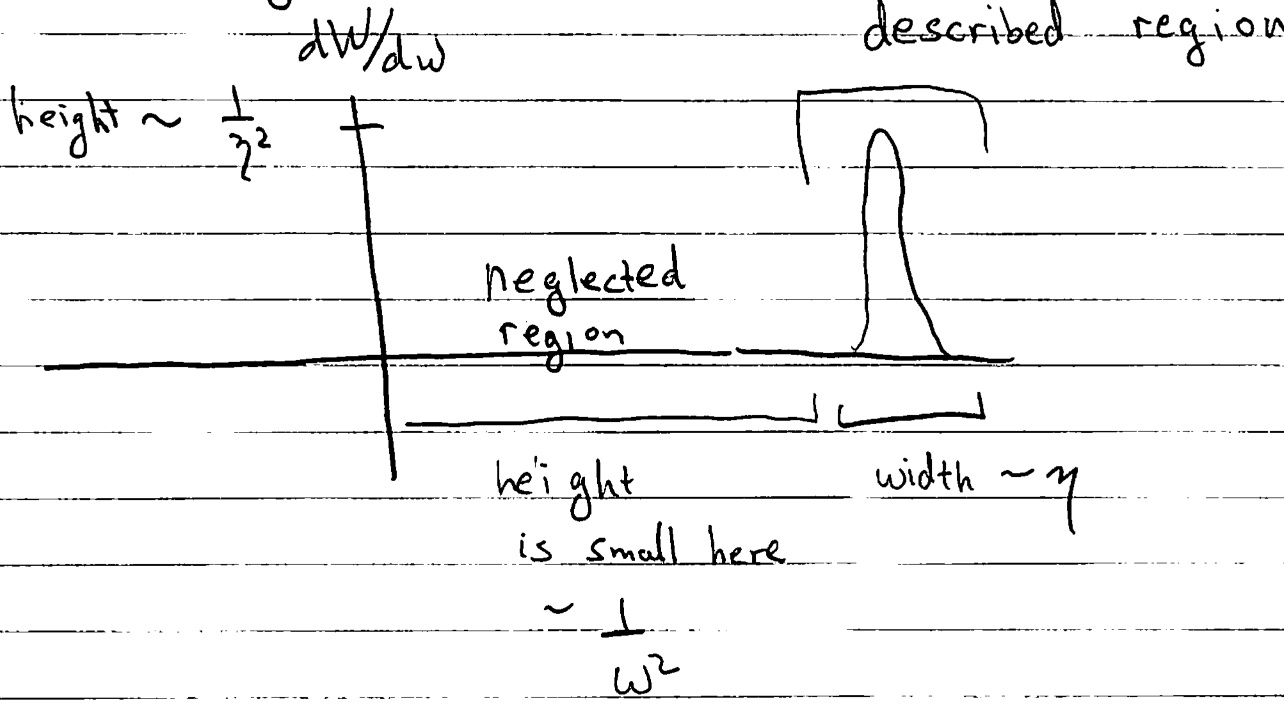
(You can also just do the integral and expand for $\omega_0 \gg \eta$)

So

$$\boxed{W = \frac{q^2}{4\pi c^3} \left(\frac{2}{3}\right) (H\omega_0^2)^2 / 2\eta}$$

in agreement with before

e) Esimating the error



The integral from the described region is

$$I \propto \text{height} \times \text{width} \sim \frac{1}{\eta}$$

The integral from the neglected region is

$$I \propto \text{height} \times \text{width} \sim \frac{1}{\omega_0^2} \times \omega_0 \sim \frac{1}{\omega_0}$$

So the described region is $O\left(\frac{\omega_0}{\eta}\right)$ larger than the neglected region