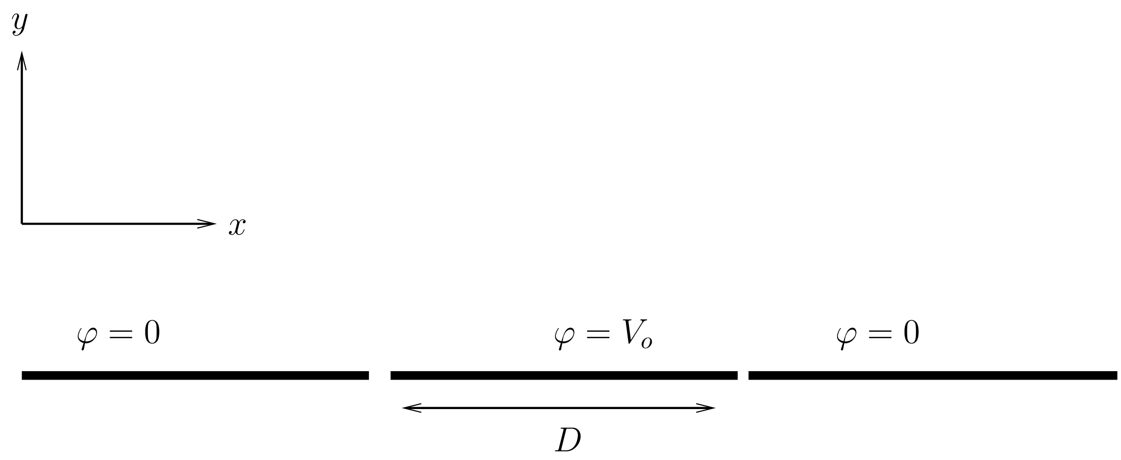


Problem 1. Potential from a strip.

An infinite conducting strip of width D (between $0 < x < D$) is maintained at potential V_o, while on either side of the strip are grounded conducting planes. The strip and the planes are separated by a tiny gap as shown below.

(a) Following a similar example given in class, determine the potential everywhere in the upper half plane $y > 0$.

(b) Determine the surface charge density on the strip and on the grounded planes, and make a graph.

Problem 2. The electric stress tensor

Recall that the stress tensor is the force per area. The force per volume f^j is (minus) the divergence of the stress tensor (see class notes)

$$f^j = -\partial_i T^{ij} \tag{1}$$

This follows from the conservation law

$$\partial_t g^j + \partial_i T^{ij} = 0 \tag{2}$$

where g^j is the momentum per volume, and the basic notion that the force is the time derivative of the momentum.

The force per volume in electrostatics is

$$f^j = \rho E^j \tag{3}$$

This form must be the divergence of something. As you will show in this excercise

$$\rho E^j = -\partial_i T_E^{ij} \tag{4}$$

where

$$T_E^{ij} \equiv -E^i E^j + \tfrac{1}{2} E^2 \delta^{ij} \tag{5}$$

(a) (Optional) First write the electrostatic Maxwell equations $\nabla \cdot \boldsymbol{E} = \rho$ and $\nabla \times \boldsymbol{E} = 0$ using tensor notation, and explain why $\partial_i E_j = \partial_j E_i$.

(b) Within the limits of electrostatics, show that the electric force on a charged body is related to a surface integral of the (electric) stress tensor:

$$F^j = \int_V d^3\boldsymbol{r}\, \rho(\boldsymbol{r})\, E^j = -\int_S dS\, n_i T_E^{ij} \tag{6}$$

where $T_E^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$, i.e. show that $\rho E^j = -\partial_i T_E^{ij}$

Problem 3. A stress tensor tutorial

Do not turn in the optional parts.

(a) (Optional) Consider a plane of charge with surface charge density σ, use the boundary conditions (i.e. Gauss Law) to show that the electric field on either side is $\sigma/2$

(b) (Optional) Consider an ideal infinite parallel plate capacitor with surface charge densities σ and $-\sigma$ respectively. Without using the stress tensor machinery, show that the force per area on each of the plates is $\sigma^2/2$

(c) (Optional) Consider a charged perfectly conducting solid object of any shape. Explain physically why the electric field is: (i) normal to the surface, (ii) zero on the inside, (iii) and equal to

$$\boldsymbol{E} = \sigma\,\boldsymbol{n} \qquad \text{or} \qquad E^i = \sigma\, n^i \tag{7}$$

(d) Without using the stress tensor machinery, show that the force per area on the walls of any metal surface is $\sigma^2/2$. (*Hint:* how large is the self field? Use part (a).)

The physics of the stress tensor is easy illustrated by knowing that the stress tensor of ideal gas is $T^{ij}_{\rm gas} = p\,\delta^{ij}$, where p is the pressure (force per area). Thus, if one considers a wall separating two gasses of left and right pressures p_L and p_R (*i.e.* the normal vector is[1], $n^j = \delta^{jx}$), then the net force per area on the wall is

$$n_i T^{ij}_L - n_i T^{ij}_R = (p_L - p_R)\, n^j \tag{8}$$

Note: that it is only the differences in the stress tensor which are physically important.

(e) (Optional) Recall that the net force on any object

$$F^j = -\oint dS\, n_i T^{ij}\,, \tag{9}$$

which we derived from the conservation law

$$\partial_t g^j + \partial_i T^{ij} = 0\,. \tag{10}$$

Deduce from this that the net force per area on a wall separating two regions is

$$n_i (T^{ij}_{\rm out} - T^{ij}_{\rm in})\,. \tag{11}$$

(f) Using the electric stress tensor $T^{ij}_E = -E^i E^j + \frac{1}{2}E^2\delta^{ij}$, show that the force per area on the surface of a charged metal object is

$$\text{force-per-area} = \frac{\sigma^2}{2} n^j \tag{12}$$

where $\boldsymbol{n}$ points from inside the metal to out.

[1] The notation is to confuse/educate you – I could have written $\boldsymbol{n} = (1, 0, 0)$ or $\boldsymbol{n} = \hat{\boldsymbol{x}}$.

(g) Now consider a charged and isolated parallel plate capacitor with charge per area $-\sigma$ and $+\sigma$ on the left and right plates (so that the normal is $n^j = \delta^{jx}$). A plane of charge with charge per area $\sigma/2$ lies halfway between the plates.

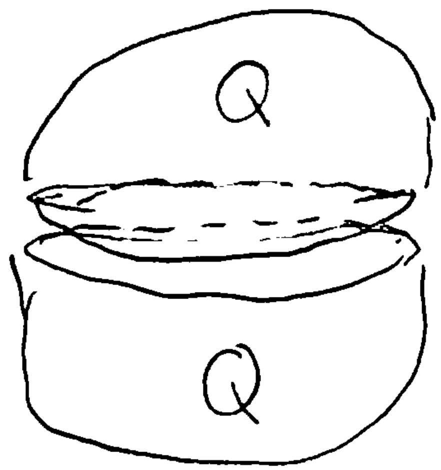
(i) Compute all non-zero components of the stress tensor in the regions to the left and right of the plane of charge.

(ii) Use the stress tensor to compute the force per area on the plane of charge, and show that it agrees with a simple minded approach.

Problem 4. Practice with the stress tensor

(a) Calculate the force between two (solid and insulating) uniformly charged hemispheres each with total charge Q and radius R that are separated by a small gap as shown below. You should find

$$F = \frac{3Q^2}{16\pi R^2} \tag{13}$$



Problem 5. Green function of a sphere

Consider a grounded, metallic, hollow spherical shell of radius R. A point charge of charge q is placed at a distance, a, from the center of the sphere along the z-axis. For simplicity take $a > R$.

(a) Start by momentarily setting $R = 1$, and therefore measure all lengths in units of R. a is then shorthand for a/R in this system of units. With these units, show that the distance from the point $\boldsymbol{r} = a\,\hat{\mathbf{z}}$ to any point, $\boldsymbol{n}$, on the surface of the sphere is equal (up to a constant factor of a) to the distance from a point at $\boldsymbol{r} = (1/a)\,\hat{\mathbf{z}}$ to the same point $\boldsymbol{n}$ on the sphere.

(b) Use the result of part (a) to construct the Green function of the grounded sphere of radius R using images, *i.e.* find the potential due to a point charge at $\boldsymbol{r} = a\hat{\mathbf{z}}$ in the presence of a grounded sphere.

(c) Compute the surface charge density, and show that it is correct by directly integrating to find the total induced charge on the sphere of part (b). You should find that the total induced charge is equal to the enclosed image charge (why?). Please do not use Mathematica to do integrals.

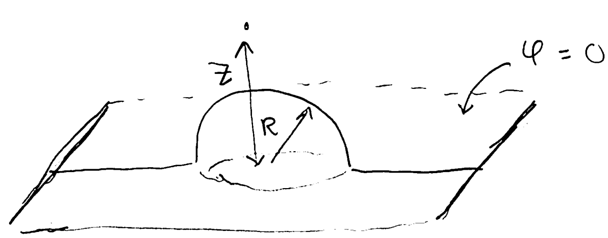
(d) Now consider a point charge of charge q at a point $\boldsymbol{r} = z\hat{\mathbf{z}}$ above a metallic hemisphere of radius R in contact with a grounded plane (see below). Determine the force on the charge as a function of z. You should find that at a distance $z = 2R$ the force is

$$F^z = -\frac{Q^2}{4\pi R^2}\left(\frac{737}{3600}\right) \tag{14}$$

(e) Show that at large distances, z, the Taylor series expansion for F^z is

$$F^z \simeq \frac{Q^2}{4\pi R^2}\left[\frac{-1}{4u^2} - \frac{4}{u^5} + \ldots\right]$$

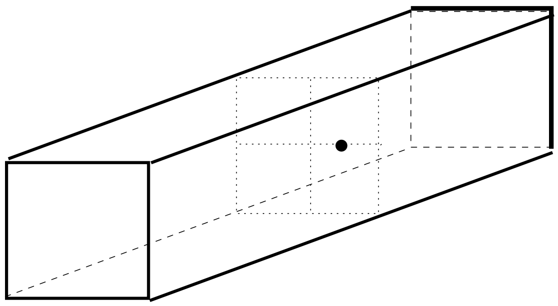
where $u = z/R$. Explicitly explain the coefficients of the series expansion (*i.e.* the $-1/4$ and -4) in terms of the multiple moments of the image solution.

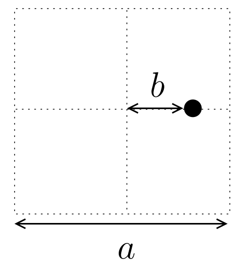


Do not hand in optional parts!

Problem 6. A point charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.





(a) (Optional) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

$$\Phi(k_x z)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \Big\{\cos(u) \text{ or } \sin(u) \tag{15}$$

for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

(b) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential. You will need an infinite number of image charges of both sign

(c) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (a) to determine the potential from the point charge described in part (b). The solution is takes the form

$$\varphi(\boldsymbol{r};\boldsymbol{r}_o) = \sum_{n,m} X_n(x)X_n(b)\,Y_m(y)Y_m(0)\frac{e^{-\kappa_{n,m}|z|}}{2\kappa_{n,m}} \tag{16}$$

(d) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$. You should find that the force (per area) on the bottom plate is

$$\frac{F^y}{A} = \frac{q^2}{a^4}\cos^2(\pi x/a)\cos^2(\pi b/a)\,e^{-2\sqrt{2}\pi|z|/a}\,. \tag{17}$$