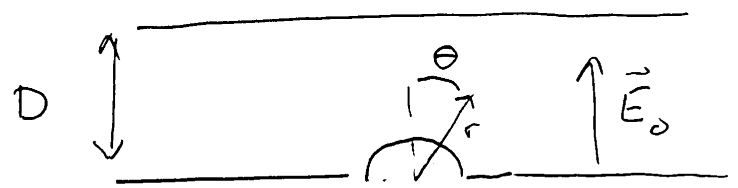
Problem 1. Defects

This problem will study defects in parallel plate capacitors. A parallel plate capacitor has area, A, and separation, D, and is maintained at the potential difference, $\Delta V = E_o D$. There are n defects per unit area on the lower plate and none on the upper. The defects consist of hemispherical shells of radius a bending towards the upper plate. You should assume that $a \ll D$, and that $na^2 \ll 1$ so that the defects are very widely spaced.



(a) Determine the charge per unit area on and near the defect. Plot the surface charge on the hemisphere as a function of θ, and on the plane as a function of r. (Hint: To solve for the potential in the vicinity of a defect use that fact that for $a \ll z \ll D$ the potential reaches its unperturbed form $\Phi(z) = -E_o z$, so that the upper boundary can be ignored.)

(b) Show that the charged induced on the hemisphere is:

$$Q = E_o a^2 \, 3\pi \tag{1}$$

(c) Use these results to show that the capacitance is unchanged by the defect to the order we are working, *i.e.*

$$C \simeq \frac{A}{d} \tag{2}$$

(d) In deriving this result we have used that $D \gg a$. The size of corrections to the potential you found are of order $\sim a^3/D^3$. Explain why.

Problem 5. Defects

This problem will study defects in parallel plate capacitors. A parallel plate capacitor has area, A, and separation, D, and is maintained at the potential difference, $\Delta V = E_o D$. There are n defects per unit area on the lower plate and none on the upper. The defects consist of hemispherical shells of radius a bending towards the upper plate. You should assume that $a \ll D$, and that $na^2 \ll 1$ so that the defects are very widely spaced.

Figure: A defect on a capacitor plate.

(a) Determine the charge per unit area on and near the defect. Plot the surface charge on the hemisphere as a function of θ, and on the plane as a function of r. (Hint: To solve for the potential in the vicinity of a defect use that fact that for $a \ll z \ll D$ the potential reaches its unperturbed form $\varphi(z) = -E_o z$, so that the upper boundary can be ignored.)

(b) Show that the charged induced on the hemisphere is:

$$Q = E_o a^2 \, 3\pi \tag{55}$$

(c) Use these results to show that the capacitance is unchanged by the defect to the order we are working, *i.e.*

$$C \simeq \frac{A}{d} \tag{56}$$

(d) In deriving this result we have used that $D \gg a$. The size of corrections to the potential you found are of order $\sim a^3/D^3$. Explain why.

Solution:

Part (a)

We write

$$\begin{aligned} \varphi =& - E_o z + \Delta\phi \quad (57) \\ =& - E_o a \cos\theta + \Delta\phi \quad (58) \end{aligned}$$

Since on the hemisphere the potential $\phi = 0$ we must require that

$$\Delta\phi = E_o a \cos\theta \qquad \text{on hemisphere} \tag{59}$$

and $\Delta\phi = 0$ on the plane.

We can introduce a series expansion

$$\Delta\phi = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{60}$$

The fact that $\Delta\phi = 0$ on the plane means that $A_\ell = B_\ell = 0$ for ℓ even. For ℓ odd, requiring regularity leads to $A_\ell = 0$. B_ℓ is adjusted to reproduce the boundary conditions as $r = a$:

$$\Delta\phi\big|_{r=a} = E_o a \cos\theta = \sum_\ell \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) \tag{61}$$

Recognizing that $P_1(\cos\theta) = \cos\theta$ we compare the two series and find $B_1 = E_o a(a^2)$ leading to

$$\phi(r,\theta) = -E_o r\cos\theta + E_o a \left(\frac{a}{r}\right)^2 \cos\theta \tag{62}$$

Then we determine the charge on the sphere

$$E_r = -\partial_r\phi = E_o\cos\theta + 2E_o a^3 \frac{1}{r^3}\bigg|_{r=a} \tag{63}$$

$$= 3E_o\cos\theta \tag{64}$$

Then we also determine the charge density on the plane

$$E_z = -E_\theta = -\left(-\frac{1}{r}\partial_\theta\phi\right) \tag{65}$$

$$= E_o - E_o\frac{a^3}{r^3} \tag{66}$$

$$\tag{67}$$

Part (b)

We integrate the surface density on the sphere , with $x = \cos\theta$

$$Q_{\text{sphere}} = \int_{\text{hemisphere}} da\,\sigma \tag{68}$$

$$Q = 3E_o a^2 2\pi \int_0^1 dx x \tag{69}$$

$$Q_{\text{sphere}} = E_o a^2 3\pi \tag{70}$$

We similarly find the charge on the plane

$$Q_{\text{plane}} = \int_a^\infty (2\pi r)dr \left(E_o - \frac{E_o a^3}{r^3}\right) \tag{71}$$

$$= E_o(A - \pi a^2) - 2E_o a^2 \tag{72}$$

$$= E_o A - 3\pi E_o a^2 3\pi \tag{73}$$

Part (c)

The total charge is unchanged:

$$Q_{\text{tot}} = (E_o a^2 3\pi) + E_o A - (E_o a^2 3\pi) = E_o A \tag{74}$$

Thus the charge and the capacitance is unchanged

$$Q_{\text{tot}} = C\Delta V \Rightarrow C = \frac{A}{d} \tag{75}$$

Part (d)

At the upper plate the potential should be constant. But it is not. The defect violates the boundary condition on the upper plate by a small amount, which can be estimated by comparing the size of the first and second terms of the potential:

$$\phi(r,\theta) = -E_o r\cos\theta + E_o a\left(\frac{a}{r}\right)^2 \cos\theta \tag{76}$$

The relative size of these two terms at the upper plate, where $r \sim D$ is of order $\sim a^3/D^3$

Problem 2. Force between two rings of charge

A single ring of charge of radius a and total charge Q is centered at the origin and lies in the xy plane.

(a) Show that the potential far from the ring can be written as the multipole expansion

$$\Phi = \frac{Q}{4\pi} \sum_{\ell} \frac{a^{\ell} P_{\ell}(0)}{r^{\ell+1}} P_{\ell}(\cos\theta) \tag{3}$$

$$\simeq \frac{Q}{4\pi r} - \frac{1}{2}\frac{Qa^2}{4\pi r^3} P_2(\cos\theta) + \frac{3}{8}\frac{Qa^4}{4\pi r^5} P_4(\cos\theta) + \ldots \tag{4}$$

where θ is measured relative to the z axis, and were in the second line we have used the known values for $P_{\ell}(0)$. What are the values of the spherical multipoles $q_{\ell m}$ (up to $\ell = 2$), and the cartesian multipoles p_i and $\mathcal{Q}_{ij}$.

(b) For a ring of charge of radius a, use an elementary argument to determine the potential along the z axis. Verify that it agrees with the expansion of part (a) when part (a) is evaluated on the z axis.

(c) Show that the force between two coaxial charged rings of charge Q and $-Q$ widely separated by a distance, $2Z$, along the z axis is

$$F \simeq \frac{-Q^2}{16\pi Z^2} + 3\frac{Q^2 a^2}{64\pi Z^4} + \ldots \tag{5}$$

where a negative answer indicates an attractive force.

An elegant way to find this is to use the Green Reciprocity theorem which is equivalent to the statement that $G_D(\boldsymbol{r}, \boldsymbol{r}_0) = G_D(\boldsymbol{r}_0, \boldsymbol{r})$. In this context, use this condition to show that the potential energy of a quadrupole charge distribution in the electrostatic potential from a monopole is the same as the potential energy of a monopole in an electrostatic potential from a quadrupole.

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Problem - Force between two rings

a)

Using the multipole expansion

(☆) $$\varphi(\vec{r}) = \sum_{lm} \frac{1}{2l+1} q_{lm} Y_{lm}(\theta,\phi) / r^{l+1}$$

Where

$$q_{lm} = \int d^3\vec{r}\, \rho(\vec{r})\, r^l Y^*_{lm}$$

Using

$$\rho(\vec{r}) = \frac{\lambda}{r} \delta(r-a)\, \delta(\cos\theta - 0)$$

We find

$$q_{lm} = \int r^2\, d(\cos\theta)\, d\phi\, \frac{\lambda}{r} \delta(r-a)\, \delta(\cos\theta - 0)\, Y^*_{lm} r^l$$

$$\boxed{q_{lm} = 0 \quad \text{for } m \neq 0}$$

Since $\int d\phi\, Y_{lm} = 0$ for $m \neq 0$

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$$q_{\ell 0} = \int r^2\, d(\cos\theta)\, \frac{\lambda}{r}\, \delta(r-a)\, \delta(\cos\theta - 0) \times \underbrace{\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)\, r^\ell}_{Y_{\ell 0}}$$

$$q_{\ell 0} = a^{\ell+1} \lambda \underbrace{2\pi}_{\text{from } \int d\phi} \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(0)$$

$$\boxed{q_{\ell 0} = Q a^\ell \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(0)}$$

Substituting these results into Eq ☆ on previous page

$$\varphi(r) = \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} Q a^\ell \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(0) \sqrt{\frac{(2\ell+1)}{4\pi}}\, P_\ell(\cos\theta)$$

$$\boxed{\varphi(r) = \frac{Q}{4\pi} \sum_{\ell=0}^{\infty} \frac{a^\ell}{r^{\ell+1}} P_\ell(0)\, P_\ell(\cos\theta)}$$

Using the known values of $P_\ell(0)$ which can derived using part b) of this proble

$$\varphi = \frac{Q}{4\pi r}\left(1 - \underbrace{\frac{1}{2}}_{P_2(0)} \frac{a^2}{r^2} P_2(\cos\theta) + \underbrace{\frac{3}{8}}_{P_4(0)} \frac{a^4}{r^4} P_4(\cos\theta) + \dots\right)$$

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Then to determine Q_{ij} we compare this series to the multipole expansion:

$$\varphi = \frac{Q}{4\pi r}\left(1 + \frac{\vec{p}\cdot\hat{r}}{r} + \frac{1}{2}Q_{ij}\frac{\hat{r}^i\hat{r}^j}{r^2} + \dots\right)$$

Comparison shows $\vec{p} = 0$, and that

$$\frac{1}{2}Q_{ij}\,\hat{r}^i\hat{r}^j = -\frac{1}{2}a^2 P_2(\cos\theta).$$

or

$$\frac{1}{2}Q_{ij}\,\hat{r}^i\hat{r}^j = -\frac{1}{2}a^2\,\frac{1}{2}(3x^2-1) \qquad x \equiv \cos\theta$$

Taking Q_{ij} to be diagonal with only Q_{zz} & Q_{xx} & Q_{yy}

$$Q_{ij} = \begin{pmatrix} -\frac{Q_{zz}}{2} & & \\ & -\frac{Q_{zz}}{2} & \\ & & Q_{zz} \end{pmatrix} \leftarrow \text{traceless}$$

So that with $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

$$\frac{1}{2}Q_{ij}\,\hat{r}^i\hat{r}^j = \frac{1}{2}\Big(\underbrace{-\frac{1}{2}Q_{zz}\sin^2\theta}_{= Q_{xx}\hat{r}^x\hat{r}^x + Q_{yy}\hat{r}^y\hat{r}^y} + \underbrace{Q_{zz}\cos^2\theta}_{= Q_{zz}\hat{r}^z\hat{r}^z}\Big)$$

$$= \frac{1}{2}Q_{zz}\frac{1}{2}(3x^2-1)$$

gives

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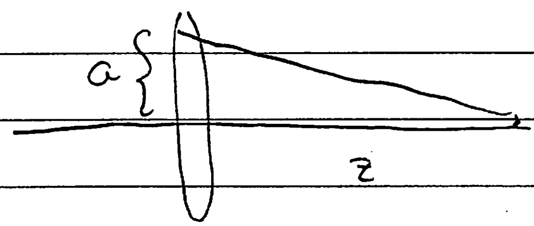
Find

$$Q_{zz} = -a^2 \qquad Q_{xx} = Q_{yy} = \frac{a^2}{2}$$

b)



$$\varphi(r) = \int \frac{\lambda \, d\phi \, a}{(a^2+z^2)^{1/2} \, 4\pi}$$

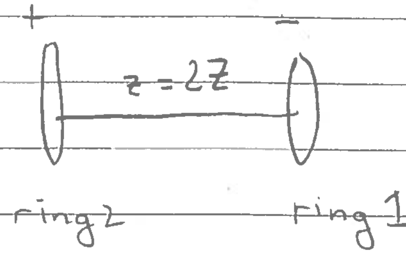
$$= \left(\frac{Q}{2\pi a}\right)(2\pi a) \frac{1}{4\pi \sqrt{a^2+z^2}}$$

$$= \frac{Q}{4\pi (z^2+a^2)^{1/2}}$$

$$\simeq \frac{Q}{4\pi z}\left(1 - \frac{1}{2}\frac{a^2}{z^2} + \underset{\uparrow \atop \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}}{\frac{3}{8}}\frac{a^4}{z^4} + \cdots\right)$$

Agrees w/ before

Rings - Part c pg. 1



c) $$U = \frac{1}{2}\int_r \rho(r)\,\varphi(r)\, d^3r$$

integrates over ring 1 and ring 2

$$U = \int_{\substack{\text{ring 1}\\ \text{only}}} \rho(r)\,\varphi(r)$$

potential from ring 2

Since ring 1 is small the results of the multipole expansion apply

field from 2

$$U = -Q\,\varphi(r) - \frac{1}{6} Q^{ij}\,\partial_i E_j$$

quadrapole of negative ring ring 1

Where

$$\varphi(r) \equiv \varphi^{mono} + \varphi^{quad} + \ldots$$

$$= \underbrace{\frac{Q}{4\pi z}}_{mono} \underbrace{- \frac{1}{2}\frac{Q}{4\pi z^3} a^2 P_2(1)}_{\equiv \varphi^{quad}}$$

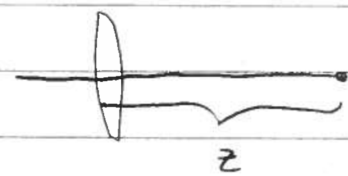
Then

$$U = \left(-\frac{Q^2}{4\pi z} \overbrace{- Q\,\varphi^{quad}}^{Q^{mono}\varphi^{quad}}\right) + \overbrace{\left(-\frac{1}{6} Q^{ij}\,\partial_i E_j^{mono}\right)}^{\equiv Q^{quad}\varphi^{mono}}$$

$$+ \text{small}^2$$

Rings. part c pg. 2

$$E_z^{mono} = \frac{Q}{4\pi z^2} \Big|_{\text{on-axis}}$$



$$\partial_z E_z = \frac{-Q}{4\pi z^3} \cdot 2$$

Now, $\partial_x E_x + \partial_y E_y + \partial_z E_z = 0$ and $\overbrace{\partial_x E_x = \partial_y E_y}^{\text{Symmetry}}$

So $\partial_x E_x = \partial_y E_y = \dfrac{+Q}{4\pi z^3}$.

Then, $-\dfrac{1}{6} Q^{ij} \partial_i E_j = -\dfrac{1}{6} \left[Q^{xx} \partial_x E_x + Q^{yy} \partial_y E_y + Q^{zz} \partial_z E_z \right]$

quadropole of negative ring.

$$= +\frac{1}{2} \frac{a^2 Q^2}{4\pi z^3} \equiv Q^{quad} \varphi^{mono}$$

Similary we have $Q^{mono} \varphi^{quad} = \dfrac{+Q^2 a^2}{2(4\pi) z^3}$

These agree -- Green Reciprocity

So

$$U_{int} = \frac{-Q^2}{4\pi z} + \cancel{2} \left(\frac{1}{\cancel{2}} \frac{Q^2}{4\pi} \frac{a^2}{z^3} \right)$$

$$= \frac{-Q^2}{4\pi z^2} + \frac{Q^2 a^2}{4\pi z^3}$$

Then differentiating

$$F^z = -\partial_z U = -\frac{Q^2}{4\pi z^2} + \frac{Q^2 a^2}{4\pi z^3}$$

So if $z = 2Z$

$$F^z = -\frac{Q^2}{16\pi Z^2} + \frac{3Q^2 a^2}{64\pi Z^4}$$

Problem 3. A ring of charge close to a plane

(a) Consider a long *line* of charge separated from a grounded plane by separation z_o. The charge per length is λ. Determine the force per length between the grounded plane and the charged line.

(b) By integrating the force found in part(a), show that the potential energy per length of the line of charge and the grounded plane is

$$u_{\rm int} = \frac{\lambda^2}{4\pi} \log 2z_o + \text{const} \tag{6}$$

This potential energy for is exactly half of the potential energy between the line of charge and its image. Qualitatively explain why this is the case.

(c) Conser a ring of radius a and total charge Q, separated from a plane by a height z_o. Use the results of this problem to determine the total force between the ring and the plane when $z_o \ll a$. Explain qualitatively why the results of this problem apply.

Problem 4. A ring of charge close to a plane

(a) Consider a long *line* of charge separated from a grounded plane by separation z_o. The charge per length is λ. Determine the force per length between the grounded plane and the charged line.

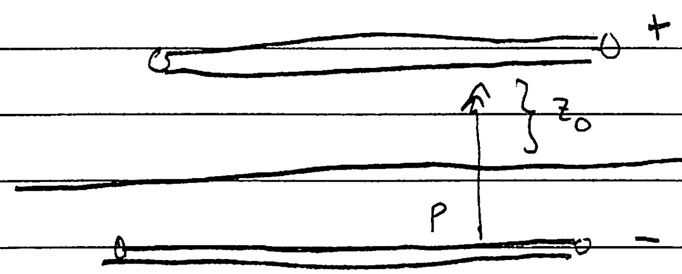
(b) By integrating the force found in part(a), show that the potential energy per length of the line of charge and the grounded plane is

$$u_{\rm int} = \frac{\lambda^2}{4\pi} \log 2z_o + \text{const} \tag{5}$$

This potential energy for is exactly half of the potential energy between the line of charge and its image. Qualitatively explain why this is the case.

(c) Conser a ring of radius a and total charge Q, separated from a plane by a height z_o. Use the results of this problem to determine the total force between the ring and the plane when $z_o \ll a$. Explain qualitatively why the results of this problem apply.

Now



$$\varphi_{line} = \frac{-\lambda}{2\pi} \log \rho + \text{const}$$

$$E_\rho^{line} = -\frac{\partial \varphi}{\partial \rho} \hat{\rho} = -\frac{\lambda}{2\pi\rho} \hat{\rho}$$

So

a) $$\frac{F}{\text{Length}} = \lambda E_\rho = -\frac{\lambda^2}{2\pi\rho} = \frac{-\lambda^2}{4\pi z_0}$$

where the negative indicates an attractive force in the negative z - direction

b) So integrating

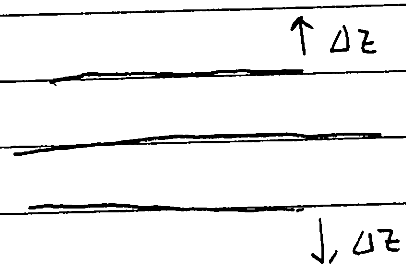
$$U_{int} = -\int_\infty^{z_0} F^z \cdot d\vec{z} = \int_{z_0}^{\infty} F^z \, dz = \int_{z_0}^{\Lambda} \frac{-\lambda^2}{4\pi z} \, dz$$

$$= -\frac{\lambda^2}{4\pi} \log \Lambda + \frac{\lambda^2}{4\pi} \log z_0$$

So

$$\mathcal{U}_{int} = \frac{\lambda^2}{4\pi} \log 2z + \text{const}$$

This energy is $\frac{1}{2}$ of the potential energy between a the line and its image

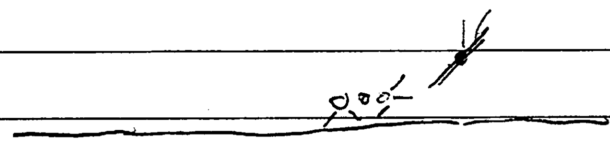


If I supply enough work to move the ring by Δz from the plane. Then the ring and its image separate by $2 \times \Delta z$.

The extra work is supplied by the source keeping the potential on the plate fixed.

c) When $z_o \ll a$,

An ant living on the plate would be unable to see the curvature of the ring.



$$F^z = -\frac{\lambda^2}{4\pi z} \cdot L$$

$$= \frac{-1}{4\pi z_o}\left(\frac{Q}{2\pi a}\right)^2 \cdot 2\pi a$$

$$\boxed{F^z = -\frac{Q^2}{8\pi^2 z_o a}}$$

Problem 4. The free green function in cylindrical coordinates

(a) Show that the green function in cylindrical coordinates can be expanded as

$$G(\boldsymbol{r}, \boldsymbol{r}_o) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} kdk \, \left[e^{im\phi} J_m(k\rho)\right] \left[e^{-im\phi_o} J_m(k\rho_o)\right] g_{km}(z, z_o) \tag{7}$$

and determine the appropriate equation for $g_{km}(z, z_o)$. (Hint: this may be a good time to examine the course notes and to write $\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) = \frac{1}{\rho}\delta(\rho - \rho_o)\delta(z - z_o)\delta(\phi - \phi_o)$ as an expansion in eigen functions in the ρ, ϕ directions)

(b) (Optional) If you dont know what a Bessel function looks like, plot $J_0(x)$, $J_1(x)$, $J_2(x)$ and record their series expansions at small and large x. Be aware of the following indentity in 2 dimensions: the 2D function

$$e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp} \tag{8}$$

can be written as a fourier series at each radius $r_\perp$. Defining $\boldsymbol{r}_\perp = r_\perp(\cos\phi, \sin\phi)$, we have

$$e^{ik_\perp r_\perp \cos\phi} = \sum_{m=-\infty}^{\infty} e^{im\phi} \, i^m J_m(kr_\perp) \tag{9}$$

(c) Consider a two dimensional function $f(r_\perp)$ which is independent of the azimuthal angle ϕ. Its Fourier transform, $\hat{f}(k_\perp) = \int d^2\boldsymbol{x}_\perp e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp}$, is independent of the azimuthal of the $\boldsymbol{k}_\perp$. Using Eq. (9), determine an integral relation between $\hat{f}(k_\perp)$ and $f(r_\perp)$ (and vice versa) using the Bessel function $J_0(k_\perp r_\perp)$. We say that $\hat{f}(k_\perp)$ and $f(r_\perp)$ are (up to a constant) Hankel transforms of each other (google Hankel transform).

(d) Use the methods discussed in class (Zangwill calls this the method of direct integration) to show that the free Green function in cylindrical coordinates can be written

$$G_o(\boldsymbol{r}, \boldsymbol{r}_o) \equiv \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} kdk \, \left[e^{im\phi} J_m(k\rho)\right] \left[e^{-im\phi_o} J_m(k\rho_o)\right] \frac{e^{-k(z_> - z_<)}}{2k} \tag{10}$$

where $z_>$ and $z_<$ is the greater and lesser of z and z_o.

It is useful to compare this result to the one derived in class

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \sum_{\ell=0}^{\infty} \sum_{-\ell}^{\ell} \left[Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta_o, \phi_o)\right] \frac{1}{2\ell + 1} \frac{r_<^\ell}{r_>^{\ell+1}} \tag{11}$$

and to a similar problem that could have been asked

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int \frac{dk}{2\pi} \left[e^{im(\phi-\phi_o)} e^{ik(z-z_o)}\right] I_m(k\rho_<) K_m(k\rho_>) \tag{12}$$

Problem 3

$$-\nabla^2 G = \delta^3(r - r_0)$$

$$-\nabla^2 G = \frac{1}{\rho}\,\delta(\rho-\rho_0)\,\delta(z-z_0)\,\delta(\phi-\phi_0)$$

Then writing:

$$G = \frac{1}{2\pi}\sum_m \int_0^\infty k\,dk\; J_m(k\rho)\, J_m(k\rho_0)\, e^{im(\phi-\phi_0)}\; g_{km}(z,z_0)$$

$$\frac{1}{\rho}\,\delta(\rho-\rho_0)\,\delta(\phi-\phi_0) = \frac{1}{2\pi}\sum_m \left[J_m(k\rho)\,e^{im\phi}\right]\left[J_m(k\rho_0)\,e^{-im\phi_0}\right]$$

Using

$$-\nabla^2 = \left[\frac{-1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{-1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{-\partial^2}{\partial z^2}\right]$$

and bessels equation + trig eqns

$$\frac{-\partial^2}{\partial\phi^2}e^{im\phi} = m^2 e^{im\phi}$$

$$\left[\frac{-1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2}\right] J_m(k\rho) = k^2 J_m(k\rho)$$

bessels eqn

We seen that

$$-\nabla^2 G = \frac{1}{2\pi}\sum_m \int_0^\infty k\,dk \left[J_m e^{im\phi}\right]\left[J_m(k\rho_0)\, e^{-im\phi_0}\right]$$

$$\left[k^2 + \frac{-\partial}{\partial z^2}\right] g_{km}(z, z_0)$$

$$= \underbrace{\frac{1}{2\pi}\sum_m \int_0^\infty k\,dk \left[J_m e^{im\phi}\right]\left[J(k\rho_0)\, e^{-im\phi_0}\right] \delta(z-z_0)}_{\frac{1}{\rho}\delta(\rho-\rho_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0)}$$

Comparison shows

a) $$\boxed{\left[+k^2 + \frac{-\partial^2}{\partial z^2}\right] g_{km}(z, z_0) = \delta(z-z_0)}$$

The two homogeneous solutions

$$y_1 = e^{kz} \quad \text{and} \quad y_2 = e^{-kx}$$

$$g(z, z_0) = C\underbrace{\left[e^{-kz} e^{kz_0}\, \Theta(z-z_0) + e^{kz} e^{-kz_0}\, \Theta(z_0 - z)\right]}_{\text{continuity}}$$

Now integrating a) for $z = z_0 - \varepsilon$ to $z_0 + \varepsilon$

$$-\left.\frac{\partial g}{\partial z}\right|_{z_0+\varepsilon} + \left.\frac{\partial g}{\partial z}\right|_{z_0-\varepsilon} = 1$$

This gives:

$$C\left[k\, e^{-kz_0} e^{kz_0} + k\, e^{kz_0} e^{-kz_0}\right] = 1$$

$$C = \frac{1}{2k}$$

So $\quad g(z, z_0) = \dfrac{e^{-kz_>} e^{kz_<}}{2k}$

And so

$$\boxed{G = \frac{1}{2\pi} \sum_m \int_0^\infty k\,dk \left[J_m(k\rho)\, e^{im\phi}\right]\left[J_m(k\rho_0)\, e^{-im\phi_0}\right] \left(\frac{e^{-k(z_> - z_<)}}{2k}\right)}$$

Problem 5. The potential energy of a charged ring

A charged ring of radius a and total charge Q is at a height z_o above a grounded plane.

(a) Show that the interaction energy between the plane and the ring is

$$U_{\text{int}}(z_o) = U(z_o) - U_{\text{self}} = \frac{1}{2}\int_{\text{ring}} d^3r \int_{\text{ring}} d^3r_1\, \rho(\boldsymbol{r})\, [G(\boldsymbol{r}, \boldsymbol{r}_1) - G_o(\boldsymbol{r}, \boldsymbol{r}_1)]\, \rho(\boldsymbol{r}_1) \quad (13)$$

where $G(\boldsymbol{r}, \boldsymbol{r}_1)$ is the green function of a point charge in the presence of the grounded plane, and $G_o(\boldsymbol{r}, \boldsymbol{r}_1)$ is the free green function.

(b) From the image solution for the Green function and the expansion given in Eq. (10), show that the interaction energy of a ring with a grounded potential is

$$U_{\text{int}}(z_o) = -\frac{Q^2}{8\pi a}\int_0^\infty dx\, [J_0(x)]^2\, e^{-2x(z_o/a)} \quad (14)$$

The last remaining integral can be done (with Mathematica)

$$U_{\text{int}}(z_o) = -\frac{Q^2}{8\pi a}\left[\frac{a}{z\pi}\,\text{EllipticK}(-\frac{a^2}{z^2})\right] \quad (15)$$

(c) Starting from the integral in Eq. (14) and the expansion of the Bessel function (see DLMF), determine the asymptotic form of the force on the ring for $z_o \gg a$. You should find that your result is in agreement with Eq. (5).

(d) When $z_0 \ll a$, use the series expansions of complete elliptic integrals available in Mathematica (`FullSimplify[Series[EllipticK[-y],...], Assumptions->{y>0}]` worked for me), to show that the potential energy the ring and the plane is:

$$U_{\text{int}}(z_o) \simeq \frac{Q^2}{8\pi^2 a}\log(z_o/4a) \quad (16)$$

Compute the force and verify consistency with Eq. (6) and its corresponding force.

(e) Use Mathematica or other program to plot the potential energy $U_{\text{int}}(z_o)/(Q^2/4\pi a)$ versus z_o/a, together with the short and long distance asymptotics all in one plot.

Problem 4

a) $$\mathcal{U}(z_0) = \frac{1}{2}\int_{ring} \rho(r)\, \varphi(r)$$

$$= \frac{1}{2}\int_{ring} \rho(r) \int_{ring} d^3r_1 \; G(r, r_1)\, \rho(r_1)$$

$$\mathcal{U}(z) = \frac{1}{2}\int_r \int_{r_1} \rho(r)\, G(r, r_1)\, \rho(r_1)$$

Now the self interaction $= \mathcal{U}_{self} = \int_r \int_{r_1} \rho(r_1) \underbrace{\frac{1}{4\pi |r - r_1|}}_{G_0(r, r_1)} \rho(r_1)$

$$\mathcal{U}(z) - \mathcal{U}_{self} = \frac{1}{2}\int_r \int_{r_1} \rho(r) \left[G(r, r_1) - G_0(r, r_1)\right] \rho(r_1)$$

b) For the image

$$G(r) = \frac{1}{4\pi |\vec{r} - \vec{r}_1|} - \frac{1}{4\pi |\vec{r} - \vec{r}_{1I}|}$$

$$\vec{r}_{1I} = (\rho, \phi, -z_0)$$

So

$$U_{int} = \frac{1}{2}\int_r\int_{r_1} \rho(\vec{r}) \left[\frac{-1}{4\pi|\vec{r}-\vec{r}_{1I}|}\right] \rho(\vec{r}_1)$$

Using

$$\rho(r) = \lambda\, \delta(\rho - a)\, \delta(z - z_0) \qquad d^3r = \rho\, d\rho\, dz\, d\phi$$

$$U_{int} = \frac{1}{2}\int d\phi\, d\phi_1\, (\lambda a)^2 \left[\frac{-1}{4\pi|\vec{r}-\vec{r}_{1I}|}\right]$$

$$\vec{r} = (a\cos\phi,\ a\sin\phi,\ z_0)$$

$$\vec{r}_{1I} = (a\cos\phi_1,\ a\sin\phi_1,\ -z_0)$$

So:

$$\frac{-1}{4\pi|\vec{r}-r_{1I}|} = \frac{-1}{2\pi}\sum_m \int k\,dk\, [J_0(ka)\, e^{im\phi}][J_0(ka)\, e^{-im\phi_1}]\, \frac{e^{-k2z_0}}{2k}$$

With

$$\int_0^{2\pi} d\phi\, e^{im\phi} = 2\pi\,\delta_{m0}$$

only $m=0$ survives.....

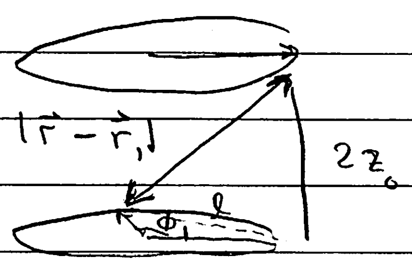
$$U_{int} = \frac{1}{2}(2\pi)^2(\lambda a)^2 \frac{-1}{2\pi}\int_0^\infty k\,dk\; J_0(ka)\,J_0(ka)\,\frac{e^{-k2z_0}}{2k}$$

Cleaning up, with $Q = \lambda a 2\pi$ $\quad x \equiv ka$

$$\boxed{U_{int} = \frac{-Q^2}{8\pi a}\int_0^\infty (J_0(x))^2\, e^{-2x(z/a)}}$$

Alternatively leave ϕ fixed $= 0$:

$$\frac{-1}{4\pi|\vec{r}-\vec{r}_1|} = \frac{-1/4\pi}{(2z_0)\sqrt{1 + \frac{a^2}{z^2}\sin^2\frac{1}{2}\theta}}$$ ← general form leading to elliptic integrals



$$|\vec{r}-\vec{r}_1| = \left[(2z_0)^2 + 2a^2\overbrace{(1-\cos\theta)}^{2\sin^2(\theta/2)}\right]^{1/2}$$

So it is plausable that

$$U_{int} = \frac{-Q^2}{8\pi a}\left[\frac{a}{z\pi} K\left(\frac{-a^2}{z^2}\right)\right]$$

c) From the integral, we see that for

$z \gg a$ $\quad e^{-2x(z/a)}$ is very small <u>unless</u>

x is very small. Expanding J_0^2 term by term

$$J_0 \simeq 1 - \frac{x^2}{4} + \dots$$

$$J_0^2 \simeq 1 - \frac{x^2}{2} + \dots$$

and integrating find

$$U_{int} = -\frac{Q^2}{8\pi a}\int_0^\infty \left(1 - \frac{x^2}{2} + \dots\right) e^{-2x z_0/a}\, dx$$

Switching to $u = 2x(z_0/a)$ and using

$$\int_0^\infty e^{-u} u^n = n! = \Gamma(n+1)$$

$$U_{int} = \frac{-Q^2}{16\pi z_0} + \frac{Q^2 a^2}{64\pi z^3} + \dots$$ agreeing with before

Now

$$-\frac{\partial u_{int}}{\partial z} = \frac{-Q^2}{16\pi z_0^2} + \frac{3 Q^2 a^2}{64\pi z_0^4}$$ ✓

d) Using the elliptic integral for

$$K(-y^2) \xrightarrow[y\to\infty]{} \frac{1}{y}\log(4y) + O\left(\frac{1}{y^2}\right)$$

We find

$$u_{int} \longrightarrow \frac{-Q^2}{8\pi a}\left(\log \frac{4a}{z}\right)\frac{1}{\pi}$$

$$= \frac{Q^2}{8\pi^2 a} \log \frac{z}{4a}$$

In agreement with Problem 2

Potential Energy between a ring and a plane at height z_0

$U/(Q^2/8\pi a)$