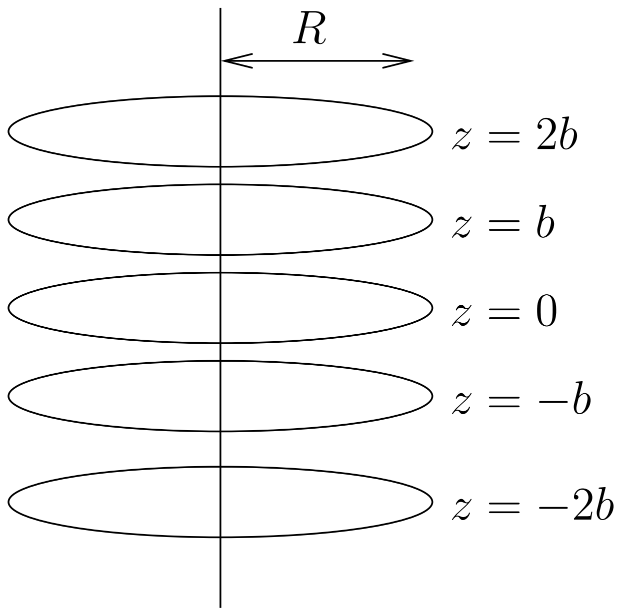
Problem 1. A Periodic Array of Charged Rings

Consider a periodic array of charged rings of radius R and separation b, so that the z-coordinates of the rings are $z = 0, \pm b, \pm 2b, \ldots$. Each ring has charge Q. We will find the potential below



(a) First classify the homogeneous solutions to the Laplace equation in cylindrical coordinates with azimuthal symmetry using separation of variables. Show that a typical homogeneous solution can be written $\varphi(z, \rho) = R(\rho)Z(z)$ and determine the equations that $R(\rho)$ and $Z(z)$ satisfy. You should find

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + k^2 R \right] = 0 \tag{1}$$

(i) What are the solutions to the $Z(z)$ equation? For z-periodic functions with period b what are the allowed values of k

(ii) Where are the singular points of radial differential equation, Eq. (1)?

(iii) (Do not hand this in) The two solutions to Eq. (1) are the modified Bessel functions $I_0(k\rho)$ and $K_0(k\rho)$ for $k^2 > 0$. Look up these functions and make a graph of them. Record (from the internet or Mathematica) a series expansion at $x = 0$ and $x = \infty$ for these functions. Note the following:

i. At $x = 0$, one function is regular and one function is irregular.

ii. At $x = \infty$ the two solutions exchange roles, with the regular function at $x = 0$ becoming irregular at $x = \infty$ and the regular function at $x = \infty$ becoming irregular at $x = 0$.

Why is this the expected behavior?

(iv) For the differential equation

$$\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]y(x)=0 \tag{2}$$

where $p(x)$ is positive definite, show that the $p(x)W(x)$ is constant, where $W(x)$ is the Wronskian of the two solutions to the differential equation. Determine $I_0'(x)K(x) - K'(x)I_0(x)$ up to a constant. Determine the (conventional) constant by using the series expansion for the modified Bessel functions at $x = 0$.

To summarize we have shown that for azimuthally symmetric functions the solutions to the homogeneous Laplace equation take the form

$$\varphi(z,\rho)=\sum_n \left(A_n I_0(k_n x)+B_n K_0(k_n x)\right)e^{ik_n x}\,, \tag{3}$$

where for periodic functions only discrete values of k are allowed. At this point you should basically understand the course notes – Appendix D.3, which treated the case where the potential vanished at $z = 0$ and $z = L$, and also allowed for non-azimuthally symmetric potentials. Now return to the problem at hand – the periodic array of rings.

(b) This problem is solved by exploiting the periodic nature of the problem, writing the charge density and the potential as a Fourier series. Use completeness to show that that the charge density is

$$\rho(\boldsymbol{x})=\frac{Q}{2\pi R}\delta(\rho-R)\frac{1}{b}\sum_{n=-\infty}^{\infty}e^{ik_n z} \tag{4}$$

where $k_n = 2\pi n/b$.

(c) Solve for the potential inside and outside the rings, and use the jump condition to relate the two solutions. Show that the potential outside of the rings is

$$\varphi(\boldsymbol{x})=\frac{Q}{2\pi b}\left[-\ln\rho+2\sum_{n=1}^{\infty}\cos(k_n z)I_0(k_n R)K_0(k_n\rho)\right] \tag{5}$$

(d) For ρ large show that

$$\varphi(\boldsymbol{x})\simeq\frac{Q}{2\pi b}\left[-\ln\rho+\sqrt{\frac{b}{\rho}}\cos\left(\frac{2\pi z}{b}\right)I_o(2\pi R/b)e^{-2\pi\rho/b}\right] \tag{6}$$

and explicitly interpret the leading term, $-\ln\rho$, and its coefficient, $Q/(2\pi b)$. Qualitatively explain the behaviour of the subleading term for large and small R/b.

Rings

a) The Laplacian

$$-\nabla^2 \varphi = 0 \qquad \text{or}$$

$$\left(-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{-\partial^2}{\partial z^2} \right)\varphi = 0$$

Substituting $\varphi = R(\rho)\,Z(z)$ into $\frac{-\nabla^2\varphi}{\varphi} = 0$

gives:

$$\underbrace{\frac{1}{R(\rho)}\frac{-1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R(\rho)} + \underbrace{\frac{-1}{Z}\frac{\partial^2}{\partial z^2}Z} = 0$$

The two terms must be constant.

$$\frac{\partial^2 Z}{\partial z^2} = -k^2 Z \qquad \left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + k^2 \right] R(\rho) = 0$$

(Eq ☆) & Eq (☆☆)

i) The solutions to $Z(z)$ equation are

$$Z_k(z) = e^{ikz}$$

Since $Z_k(z+b) = Z_k(z)$ we must have

$$k_n = \frac{2\pi n}{b} \quad \text{where} \quad n = \ldots -2, -1, 0, 1, 2 \ldots$$

ii) The singular points are at $\rho = 0$ and $\rho = \infty$

iv)
$$\left[-\frac{d}{dx} p(x) \frac{dy_1}{dx} + q(x) y_1 \right] = 0 \qquad \text{(Eq 1)}$$

and

$$\left[-\frac{d}{dx} p(x) \frac{dy_2}{dx} + q(x) y_2 \right] = 0 \qquad \text{(Eq 2)}$$

Where y_1 and y_2 are the two solutions of the differential equation. Multiply (Eq 1) by y_2 and (Eq 2) by y_1 and subtracting

$$y_2 \left(-\frac{d}{dx} p(x) \frac{d}{dx} y_1 \right) - y_1 \left(-\frac{d}{dx} p(x) \frac{d}{dx} y_2 \right) = 0$$

Or

$$-\frac{d}{dx} \left[y_2(x)\, p(x) \frac{dy_1}{dx} - y_1(x) p(x) \frac{d}{dx} y_2(x) \right] = 0$$

i.e.

$$p(x)\left[y_2 y_1' - y_1 y_2'\right] = \text{const}$$

Then comparison with Eq. 1

$$x\left[K_0(x)\, I_0'(x) - K_0'(x)\, I_0(x)\right] = \text{const}$$

Then from the series expansion

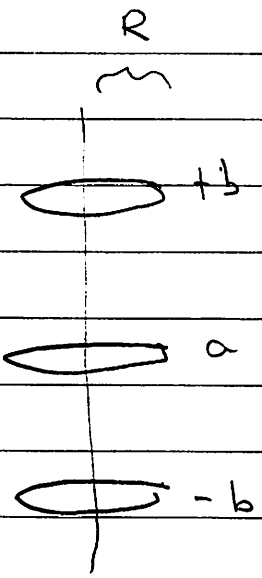
$$K_0(x) \simeq -\gamma_E + \log 2 - \log x + O(x^2)$$

$$I_0(x) \simeq 1 + O(x^2)$$

Here $\gamma_E = 0.5772$ is the Euler-Mascheroni constant.
Substituting the series we have

$$x\left[K_0 I' - K' I\right] = -1$$

Problem 6 :



$$-\nabla^2 \varphi = \lambda\, \delta(\rho - R) \sum_n \delta(z - nb)$$

- φ is periodic so we write it as a fourier series:

$$\varphi = \frac{1}{b} \sum_{n=-\infty}^{\infty} e^{ik_n z} \varphi_n(\rho) \qquad k_n = \frac{2\pi n}{b}$$

Now: $\displaystyle \sum_n \delta(z - nb) = \frac{1}{b} \sum_n e^{ik_n z}$

Recall the completeness

$$\langle n | m \rangle = b\, \delta_{nm}$$

$$\int_{-b/2}^{b/2} e^{ik_n z} e^{-ik_m z}\, dz = b\, \delta_{nm}$$

$$\frac{1}{b} \sum |n\rangle\langle n| = \mathbb{1}$$

So $-\nabla^2\varphi$ reads:

$$\left[\frac{-1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{-\partial^2}{\partial z^2}\right]\frac{1}{b}\sum_n e^{ikn}\varphi_n(\rho) = \lambda\,\delta(\rho-R)\frac{1}{b}\sum_n e^{ik_n z}$$

Or

$$(\star)\quad \left[\frac{-1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + k_n^2\right]\varphi_n(\rho) = \lambda\,\delta(\rho-R)$$

We will solve this outside and inside and match at $\rho = R$. The general solution is

$$\varphi_n(\rho) = \begin{cases} A_n I_0(k_n\rho) + B_n K_0(k_n\rho) & |k_n| > 0 \\ A_0 + B_0(-\ln\rho) & k_n = 0 \end{cases}$$

Note that $I_0 \xrightarrow[k_n\to 0]{} 1$ $\quad K(k_n\rho) \xrightarrow[k_n\to 0]{} -\ln\rho + \text{const}$

Also note for $k_n > 0$:

$$(\star\star)\quad \left[K_0(k_n\rho)\frac{\partial I_0(k_n\rho)}{\partial\rho} - I_0(k_n\rho)\frac{\partial K_0(k_n\rho)}{\partial\rho}\right] = \frac{1}{\rho}$$

And that this also holds when $k_n \to 0$: i.e.

$$(\star\star\star)\quad \left[-\ln\rho\,\frac{\partial 1}{\partial\rho} - 1\,\frac{\partial}{\partial\rho}(-\ln\rho)\right] = \frac{1}{\rho}$$

Then regularity as $\rho \to 0$

$$\varphi_n(\rho) = \begin{cases} A_n I_0(k_n \rho) & \text{(Inside)} \\ A_0 & \text{for } k_n = 0 \end{cases}$$

While outside:

$$\varphi_n(\rho) = \begin{cases} B_n K_0(k_n \rho) & \text{(Outside)} \\ B_0(-\ln \rho) & \end{cases}$$

So continuity at $\rho = R$ gives:

$$(*) \quad \left[\begin{array}{l} \varphi_n(\rho) = \begin{cases} C_n I_0(k_n R) K_0(k_n \rho) & \text{Outside} \\ C_n K_0(k_n R) I_0(k_n \rho) & \text{Inside} \end{cases} \quad k_n \neq 0 \\ \\ \quad\quad\;\; = \begin{cases} C_0 (-\ln \rho) & \text{Outside} \\ C_0 (-\ln R) & \text{Inside} \end{cases} \quad k_n = 0 \end{array} \right.$$

Now we need to look at the jump condition

Integrating Eq ☆ from $R - \varepsilon$ to $R + \varepsilon$

$$\int_{R-\varepsilon}^{R+\varepsilon} d\rho \left[-\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + k_n^2 \rho \right] \varphi_n = \int_{R-\varepsilon}^{R+\varepsilon} \lambda \rho \, \delta(\rho - R)$$

$$(**) \quad -\rho \frac{\partial \varphi_n^{out}}{\partial \rho} \Big|_{\rho = R} + \rho \frac{\partial \varphi_n^{in}}{\partial \rho} \Big|_{\rho = R} = \lambda R$$

Substituting Eq $(*)$ into Eq $(**)$

we have

$$C_n \underbrace{\left[-\rho\, I_0(k_n R) \frac{\partial K_0(k_n\rho)}{\partial\rho} + \rho\, K_0(k_n\rho) \frac{\partial I_0(k_n\rho)}{\partial\rho} \right]}_{=1} = \lambda R$$

So

$$C_n = \lambda R$$

This also holds for $n=0$. Thus

$$\varphi = \frac{1}{b}\sum_{n\neq 0} e^{ik_n z} I_0(k_n R)\, K_0(k_n\rho)\,\lambda R \qquad \underline{\underline{\text{Outside}}}$$

$$+ \frac{1}{b}\lambda R\,(-\ln\rho)$$

For $\rho \to \infty$ all $n\neq 0$ terms disappear since (See appendix C)

$$K_0(k_n\rho) \simeq \sqrt{\frac{\pi}{2k_n\rho}}\, e^{-k_n\rho} \quad \text{for } \rho\to\infty$$

We keep only the $n=\pm 1$ terms to get the first correction:

$$K_0(k_1\rho) \simeq \sqrt{\frac{\pi}{2\left(\frac{2\pi}{b}\right)\rho}}\; e^{-2\pi\rho/b} = \sqrt{\frac{b}{4\rho}}\; e^{-2\pi\rho/b}$$

And also

$$K_0(k_{-1}\rho) \simeq \sqrt{\frac{b}{4\rho}}\, e^{-2\pi\rho/b}$$

Then outside we have with $\lambda R = \frac{Q}{2\pi}$

$$\varphi = \frac{Q}{2\pi b} \underbrace{(e^{ik_1 z} + e^{-ik_1 z})}_{2\cos(2\pi z/b)} I_0(2\pi R/b) \sqrt{\frac{b}{4\rho}}\, e^{-2\pi\rho/b}$$

$$+ \frac{Q}{2\pi b}(-\ln\rho)$$

i.e.

$$\boxed{\varphi = \frac{Q}{2\pi b}\left[-\ln\rho + \sqrt{\frac{b}{\rho}}\cos\left(\frac{2\pi z}{b}\right) I_0(2\pi R/b)\, e^{-2\pi\rho/b}\right]}$$

- The first term is what you expect.
- The combination $K_0 I_0' - I_0 K_0'$ is the wronskian. From Bessels eqn:

$$\left[-\frac{d}{dx}\, x\, \frac{d}{dx} + x\right](I_0 \text{ or } K_0) = 0$$

Thus $p(x)\, W(x)$ is const, so in this case:

$$x\left[K_0 I_0' - I_0 K_0'\right] = \text{const}$$

Problem 2. A dielectric cylinder in an external field

An infinitely long dielectric cylinder of dielectric constant ϵ and radius a (centered at the origin with axis along the z axis) is placed in an approximately constant external electric field in the x direction. The external electric field contains a constant small gradient in the x direction, $\partial_x E_x \equiv E_o'$. The external potential is described by

$$\varphi_{\rm ext}(\boldsymbol{r}) = -E_o x - \frac{1}{2}E_o'\left(x^2 - y^2\right) \tag{7}$$

The gradient is small since $E_o' a \ll E_o$.

(a) (**Optional**) Separate variables in cylindrical coordinates with $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Show that the general solution to the Laplace equation takes the form

$$\varphi = A_0 + B_0 \ln\rho + \sum_{n=1}^{\infty}\left(A_n\rho^n + \frac{B_n}{\rho^n}\right)\cos(n\phi) + \sum_{m=1}^{\infty}\left(C_m\rho^m + \frac{D_m}{\rho^m}\right)\sin(m\phi) \tag{8}$$

(**Not Optional**) When I first started writing this problem, I set $\varphi_{\rm ext}(\boldsymbol{r}) = -E_o x - \frac{1}{2}E_o' x^2$, what is wrong with this?

(b) Determine the potential both inside and outside the cylinder including the first correction due to the field gradient, E_o'. I find

$$\varphi = \begin{cases} \frac{2}{1+\epsilon}\,\varphi_{\rm ext}(\boldsymbol{r}) & r < a \\ \varphi_{\rm ext}(\boldsymbol{r}) + \frac{\epsilon-1}{\epsilon+1}\left(\frac{E_0 a^2\cos\phi}{\rho} + \frac{E_0' a^4 \cos 2\phi}{2\rho^2}\right) & r > a \end{cases} \tag{9}$$

(c) Determine the surface charge induced on the cylinder including the first correction due to the field gradient. You should find

$$\sigma = 2\frac{(\epsilon-1)}{(\epsilon+1)}\left[E_0\cos\phi + E_0' a\cos(2\phi)\right] \tag{10}$$

Using the stress tensor formalism and the boundary conditions it is possible show that the force on the cylinder is

$$F^i = \int da\, \sigma\, E^i_{\rm ext} \tag{11}$$

This is quite difficult for such an intuitive result and it is suggested that you take it on faith on first pass.

(d) Show that the net force on the cylinder per unit length is

$$\frac{F}{L} = 2\pi a^2\left(\frac{\epsilon-1}{\epsilon+1}\right)E_0 E_0' \tag{12}$$

to first order in E_0'.

Solution

a) $-\nabla^2 \varphi = 0$ in 2D

where $-\nabla^2 = \frac{-1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{-1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$

- Then writing $\varphi = R(\rho)\,\Phi$, and substituting into $-\rho^2\nabla^2\varphi/\varphi = 0$ gives:

$$-\frac{1}{R}\left(\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R\right) + \frac{1}{\Phi}\left(\frac{-\partial^2\Phi}{\partial\phi^2}\right) = 0$$

So our equations are

(1) $-\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}R + m^2 R = 0$

(2) $-\frac{\partial^2}{\partial\phi}\Phi = m^2\,\Phi$

- So the periodicity of $\Phi(0) = \Phi(2\pi)$ makes this a self adjoint eigenvalue problem with eigenvalues m = integers, and eigen-fcns:

$$\Phi_m = \begin{cases}\cos m\phi \\ \sin m\phi\end{cases}$$

with a special case of $m = 0$ with $\Phi_0 = 1$

- The $R(\rho)$ equation takes the form

$$\text{☆} \qquad R_m(\rho) = C_1 \rho^m + \frac{C_2}{\rho^m} \qquad m \neq 0$$

- The general solution is then

$$\varphi = A_0 + B_0 \ln\rho + \sum_{m=1}^{\infty} \left(A_m \rho^m + \frac{B_m}{\rho^m}\right) \cos m\phi$$

$$+ \left(C_m \rho^m + \frac{D_m}{\rho^m}\right) \sin m\phi \qquad (\text{☆☆})$$

Where the $A_0 + B_0 \ln\rho$ is the $m=0$ limit of Eq ☆.

- A form

$$\varphi = -E_0 x - \frac{1}{2} E_0' x^2$$

Does not satisfy the Laplace eqn: Indeed $x^2 = (\rho\cos\phi)^2 \propto (\cos\phi)^2$, while a solution must have from Eq ☆☆

$$\varphi \propto \cos 2\phi \propto \cos^2\phi - \sin^2\phi$$

So the solution must behave as

$$x^2 - y^2$$

b) The boundary data involves

$$\varphi \xrightarrow[\rho \to \infty]{} -E_0 \rho \cos\phi - \frac{1}{2} E_0' \rho^2 \cos 2\phi$$

- So we limit solution to $\cos\phi$ and $\cos 2\phi$ terms:

$$\varphi_{in} = A_1 \rho \cos\phi + A_2 \rho^2 \cos 2\phi$$

$$+ \text{ irregular terms set to zero}$$

$$\varphi_{out} = -E_0 \rho \cos\phi - \frac{1}{2} E_0' \rho^2 \cos 2\phi$$

$$+ \frac{B_1}{\rho} \cos\phi + \frac{B_2}{\rho^2} \cos 2\phi$$

$$+ \text{ terms which don't go to zero as } \rho \to \infty$$

- To write φ_{out} we we have extracted its behavior as $\rho \to \infty$ in the first line, and then required that remaining terms go to zero as $\rho \to \infty$.

① We then demand that φ is continuous, or the boundary condition $E''_{out} - E''_{in}$ would not be satisfied.

② We also require that $\vec{D}$ be continuous, i.e

$$\vec{n}\cdot(\vec{D}_{out} - \vec{D}_{in}) = -\cancel{\sigma_{ext}}^{\,0}$$

$$-\frac{\partial\varphi}{\partial\rho}\bigg|_{\rho=a} - \varepsilon\left(-\frac{\partial\varphi}{\partial\rho}\right)\bigg|_{\rho=a} = 0$$

on the surface.

These conditions must hold in each fourier mode separately (i.e for $\cos\phi$ and $\cos 2\phi$ terms separately). From ① (continuity)

$$-A_1 a - E_0 a + \frac{B_1}{a} = 0 \qquad (☆1)$$

$$-A_2 a^2 - \frac{1}{2}E_0' a^2 + \frac{B_2}{a^2} = 0 \qquad (☆2)$$

From ②

$$\varepsilon A_1 + E_0 + \frac{B_1}{a^2} = 0 \qquad (☆3)$$

$$+2\varepsilon A_2 a + E_0' a + 2\frac{B_2}{a^3} = 0 \qquad (\star 4)$$

- Solving ($\star$1) and ($\star$3) for A_1, B_1, and ($\star$2) and ($\star$4) for $A_2 B_2$ we find

So

$$\boxed{\begin{aligned} \varphi_{in} &= \frac{2}{1+\varepsilon}\left[-E_0 \rho \cos\phi - \frac{E_0'}{2}\rho^2 \cos 2\phi\right] = \frac{2}{1+\varepsilon}\varphi_{ext}(r) \\ \varphi_{out} &= \varphi_{ext} + \frac{(\varepsilon-1)}{(\varepsilon+1)}\left[\frac{E_0 a^2 \cos\phi}{\rho} + \frac{E_0' a^4 \cos 2\phi}{2\rho^2}\right] \end{aligned}}$$

One can check that as $\varepsilon \to 1$ we have

$$\varphi_{in} = \varphi_{out} = \varphi_{ext}$$

One can aslo check that as $\varepsilon \to \infty$, which corresponds to infinite conductivity

$$\varphi_{in} \xrightarrow[\varepsilon \to \infty]{} 0$$

c) We can coumpute the Charge density

$$\sigma = -\vec{n}\cdot(\cancel{P_{out}}^{\,0} - P_{in}) \qquad P_{in} = (\varepsilon - 1)E_{in}$$

$$\sigma = -(\varepsilon - 1)\frac{\partial \varphi_{in}}{\partial \rho}$$

Here

$$\varphi_{in} = \frac{2}{1+\varepsilon}\varphi_{ext}$$

So

$$\boxed{\sigma_{in} = 2\frac{(\varepsilon-1)}{\varepsilon+1}\left(E_0\cos\phi + aE_0'\cos 2\phi\right)}$$

d) To calculate the force

$$\vec{F} = \int da\, \sigma\, \vec{E}$$

So

$$F^x = \int da \left[\sigma^{(1)} + \sigma^{(2)}\right] \overbrace{\left(E^{(1)}_x + E^{(2)}_x\right)}^{-\partial_x \varphi}$$

where

$$\sigma^{(1)} = \frac{2(\varepsilon-1)}{\varepsilon+1} E_0 \cos\phi \qquad \sigma^{(2)} = \frac{2(\varepsilon-1)}{\varepsilon+1} E_0' a \cos 2\phi$$

dipole quadrupole

$$E^{(1)} = E_0 \quad \text{and} \quad E^{(2)}_0 = E_0' \overbrace{\rho \cos\phi}^{x}$$

So let us group these terms

$$\frac{F^x}{L} = \underbrace{\int a\, d\phi}_{\text{surface}} \left[\sigma E_0 + (\sigma^{(1)} + \sigma^{(2)})\, E_0' a \cos\phi\right]$$

this gives zero upon integration because there is no net charge on the cylinder

So

$$\frac{F^x}{L} = \int a\, d\phi\, \sigma^{(1)} E_0' a \cos\phi + O\left((E_0')^2\right)$$

$(\text{small})^2$

So performing the remaing integral $\int \cos^2\phi = \pi$

$$\frac{F^x}{L} = \frac{2(\varepsilon-1)}{(\varepsilon+1)} \pi a^2 E_0 E_0'$$

Problem 3. A point charge and a semi-infinite dielectric slab

A point charge of charge q in vacuum is at the origin $\boldsymbol{r}_o = (0,0,0)$. It is separated from a semi-infinite dielectric slab filling the space $z > a$ with dielectric constant $\epsilon > 1$. When evaluating the potential for $z < a$, an image charge solution is found by placing an image charge at $z = 2a$. When evaluating the potential for $z > a$ we place an image charge at the origin. The full image solution is

$$\varphi(\boldsymbol{r}) = \begin{cases} \frac{q}{4\pi|\boldsymbol{r}|} - \frac{\beta q}{4\pi|\boldsymbol{r}-2a\hat{\boldsymbol{z}}|} & z < a \\ \frac{\beta' q}{4\pi\epsilon|\boldsymbol{r}|} & z > a \end{cases} \tag{13}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and $\beta' = (2\epsilon)/(1 + \epsilon)$

(a) Sketch a picture of the resulting electric field lines in the xz plane. Pay particular attention to the behaviour at the interface, noting Eq. (14) derived below.

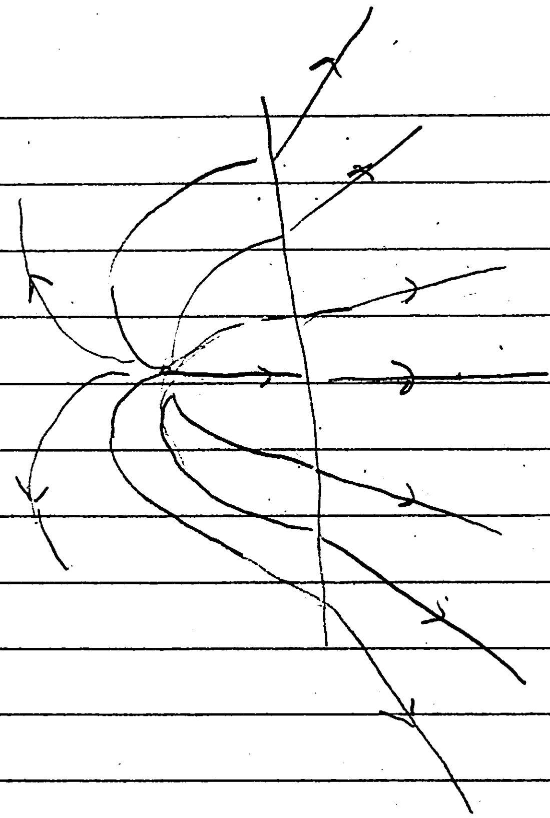
(b) Quite generally show that the electric field lines refract at a discontinuous interface

$$\frac{\tan\theta_I}{\epsilon_{\mathrm{I}}} = \frac{\tan\theta_{\mathrm{II}}}{\epsilon_{\mathrm{II}}} \tag{14}$$

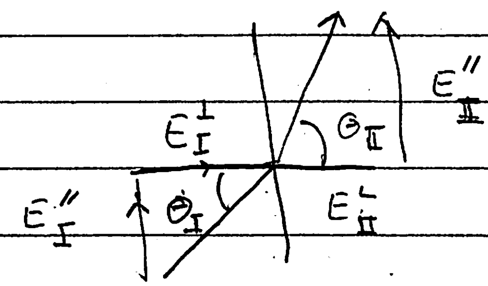
where θ_{I} and θ_{II} are the angles between the normal pointing from I to II and the electric fields in region I and region II, and ϵ_{I} and ϵ_{II} are the dielectric constants.

(5)

a)



b) Using



$$\varepsilon_{II} E_{II}^{\perp} = \varepsilon_{I} E_{I}^{\perp}$$

$$E_{II}^{\parallel} = E_{I}^{\parallel}$$

So

$$\frac{E_{II}^{\parallel}}{\varepsilon_{II} E_{II}^{\perp}} = \frac{E_{I}^{\parallel}}{\varepsilon_{I} E_{I}^{\perp}}$$

$$\frac{\tan\theta_{II}}{\varepsilon_{II}} = \frac{\tan\theta_{I}}{\varepsilon_{I}}$$

i.e. $$\boxed{\tan\theta_{II} = \frac{\varepsilon_{II}}{\varepsilon_{I}} \tan\theta_{I}}$$

Problem 4. A Dielectric slab intervenes.

This problem will calculate the force between a point charge q in vacuum and a dielectric slab with dielectric constant $\epsilon > 1$. The point charge is at the origin $\boldsymbol{r}_o = (x_o, y_o, z_o) = (0,0,0)$, but we will keep x_o, y_o, z_o for clarity. The slab lies between $z = a$ and $z = a + \delta$ with $a > 0$ and has infinite extent in the x, y directions

(a) Write the free space Green function as a Fourier transform

$$\frac{q}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} = q\int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{r}_\perp - \boldsymbol{r}_{o\perp})} g^o_{\boldsymbol{k}_\perp}(z, z_o) \tag{15}$$

and show that the free space green function in fourier space is

$$g^o_{\boldsymbol{k}_\perp}(z, z_o) = \frac{e^{-k_\perp|z-z_o|}}{2k_\perp} \tag{16}$$

(b) Now consider the dielectric slab and write the potential produced by the point charge at $z_o = 0$ as a Fourier transform

$$\varphi(\boldsymbol{r}_\perp, z) = q\int \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^2} e^{i\boldsymbol{k}_\perp \boldsymbol{r}_\perp} g_{\boldsymbol{k}_\perp}(z)\,, \tag{17}$$

and determine for $g_{k_\perp}(z)$ by solving in each region, matching across the interfaces, and by analyzing the jump at z_o. Show that for $z < 0$ and $0 < z < a$

$$g_{\boldsymbol{k}_\perp}(z) = \begin{cases} \frac{e^{kz}}{2k} - \frac{\beta e^{k(z-2a)}(1-e^{-2\delta k})}{2k(1-\beta^2 e^{-2\delta k})} & z < 0 \\ \frac{e^{-kz}}{2k} - \frac{\beta e^{k(z-2a)}(1-e^{-2\delta k})}{2k(1-\beta^2 e^{-2\delta k})} & 0 < z < a \end{cases} \tag{18}$$

where $\beta = (\epsilon - 1)/(\epsilon + 1)$ and we have written $k = k_\perp$ to lighten the notation.

(c) Checks:

(i) Show that for $\delta \to \infty$ the potential for $z < a$ is in agreement with the results of the previous problem.

(ii) Show that when $\epsilon \to \infty$ (when the dielectric becomes almost metallic) you get the right potential.

(d) Show that the electric potential for region $z < a$ can be written

$$\varphi = \varphi_{\text{ind}} + \frac{q}{4\pi r} \tag{19}$$

where φ_{ind} is the induced potential and is regular at $r = 0$. Show that the force on the point charge is

$$F^z = \beta\frac{q^2}{4\pi(2a)^2}\int_0^\infty du\, \frac{4ue^{-2u}(1 - e^{-2(\delta/a)u})}{1 - \beta^2 e^{-2(\delta/a)u}} \tag{20}$$

(e) Use a program such as mathematica to make a graph of the force $F^z/(\beta q^2/(4\pi(2a)^2))$ versus δ/a for $\beta = 0.1, 0.5, 0.9$ and sketch the result.

⑥

a) Writing

$$G_0 = \int e^{-i\vec{k}_\perp \cdot \vec{r}_\perp} \, g_k(z,z_0) \, \frac{d^2k}{(2\pi)^2}$$

Then since

$$-\nabla^2 G_0 = \delta^3(r-r_0)$$

Substituting we find

$$\int \frac{d^2k}{(2\pi)^2} e^{-ik_r} \left[\frac{-\partial}{\partial z^2} + k_\perp^2 \right] g_k(z,z_0) = \delta(z-z_0) \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k}_\perp \cdot \vec{r}_\perp}$$

$$\left[\frac{-\partial^2}{\partial z^2} + k_\perp^2 \right] g_k(z,z_0) = \delta(z-z_0)$$

The two solutions $y_1 = e^{-k_\perp z}$ $y_2 = e^{+k_\perp z}$

$W = y_1 y_2' - y_2 y_1'$

So

$$g_k(z,z_0) = \frac{e^{k_\perp z_<} e^{-k_\perp z_>}}{2k} \checkmark$$

Or

$$= \left[\frac{e^{-k_\perp z} e^{k_\perp z_0} \Theta(z-z_0) + e^{k_\perp z} e^{-k_\perp z_0} \Theta(z_0-z)}{2k} \right.$$

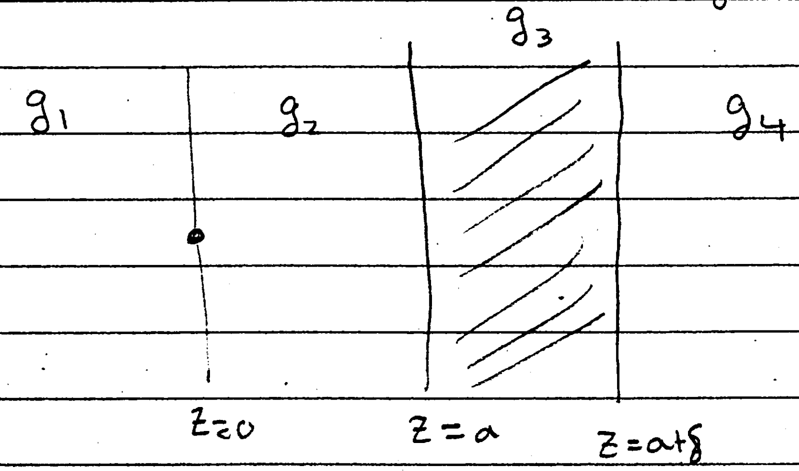
b) Write

$$\varphi(r_\perp, z) = q \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ikr} g_k$$

Then

$$\left[-\frac{\partial^2}{\partial z^2} + k_\perp^2\right] g_k = \delta(z - z_0)$$

Then we divide into regions



In each region the solution is a lin-combo or e^{-kz} and e^{kz}

$$g_4 = F e^{-kz}$$

$$g_3 = A e^{-kz} + B e^{kz}$$

$$g_2 = C e^{-kz} + D e^{kz}$$

$$g_1 = G e^{+kz}$$

Now we have

(1) $$g_4 = g_3 \Big|_{z=a+\delta}$$

(2) $$g_2 = g_3 \Big|_{z=a}$$

(3) $$g_1 = g_2 \Big|_{z=0}$$

And from continuity of $D = \frac{\partial g}{\partial z}$ at interfaces

(4) $$\frac{\partial g_4}{\partial z} = \varepsilon \frac{\partial g_3}{\partial z} \Big|_{z=a+\delta}$$

(5) $$\frac{\partial g_2}{\partial z} = \varepsilon \frac{\partial g_3}{\partial z} \Big|_{z=a}$$

Then from jump $-\frac{\partial^2}{\partial z^2} g_k + k^2 = \delta(z)$

(6) $$-\frac{\partial g_2}{\partial z}\Big|_{z=0} + \frac{\partial g_1}{\partial z}\Big|_{z=0} = 1$$

We now have six eqns and six unknowns

F, A, B, C, D, G

Plugging these six conditions into Mathematica and saying solve, we find

$$F = \frac{1-\beta^2}{2k\left(1-\beta^2 e^{-2\delta k}\right)}$$

Using Mathematica

$$g_1 = \frac{e^{kz}}{2k} + \frac{1}{2k} e^{k(z-2a)} \frac{\beta\left(-e^{-2\delta k}+1\right)}{\left(-1+\beta^2 e^{-2\delta k}\right)}$$

Now

$$\boxed{g_1 = \frac{e^{kz}}{2k} - \frac{\beta}{2k} e^{k(z-2a)} \frac{\left(1-e^{-2\delta k}\right)}{\left(1-\beta^2 e^{-2\delta k}\right)}}$$

From

$$\boxed{g_2 = \frac{e^{-kz}}{2k} - \frac{e^{k(z-2a)}}{2k} \beta \frac{\left(1-e^{-2\delta k}\right)}{\left(1-\beta^2 e^{-2\delta k}\right)}}$$

c) Checks

i) For $\delta \to \infty$

$$g_k(z) = \frac{e^{-k|z|}}{2k} - \beta \frac{e^{k(z-2a)}}{2k} \qquad \text{for } z<a$$

Fourier transforming using

$$\frac{1}{4\pi \vec{r}} = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\vec{r}} \frac{e^{-k|z|}}{2k}$$

We find

$$\varphi = q\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\vec{r}} \left(\frac{e^{-k|z|}}{2k} - \beta \frac{e^{-k|z-2a|}}{2k} \right)$$

$$\varphi = \frac{q}{4\pi|\vec{r}|} - \frac{\beta}{4\pi|\vec{r} - 2a\hat{z}|}$$

ii) For $\varepsilon \to \infty$ $\beta \to 1$

$$g = \frac{e^{-k|z|}}{2k} - \frac{e^{k(z-2a)}}{2k}$$

$$\varphi = q\int \frac{d^2k}{(2\pi)^2} \left[\frac{e^{-k|z|}}{2k} - \frac{e^{-k|z-2a|}}{2k} \right]$$

So

$$\varphi = \frac{q}{4\pi|\vec{r}|} - \frac{q}{4\pi|\vec{r} - 2a\hat{z}|}$$

iii) When $\beta \longrightarrow 0$ we have

$$\varphi = q\int \frac{d^2k}{(2\pi)^2} \frac{e^{-k|z|}}{2k}$$

$$\varphi = \frac{q}{4\pi|\vec{r}|}$$

d) Force

Examing

$$g_k = \underbrace{\frac{e^{-k|z|}}{2k}}_{\text{gives } \frac{q}{4\pi r}} - \underbrace{\frac{e^{k(z-2a)}}{2k} \beta \frac{(1-e^{-2\delta k})}{(1-\beta^2 e^{-2\delta k})}}_{\text{This is a regular function } \varphi_{ind}}$$

So

$$\varphi = q\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{-ik_\perp r_\perp}\left[\underbrace{\frac{e^{-k|z|}}{2k}}_{\substack{\text{gives} \\ \frac{q}{4\pi r}}} - \underbrace{\frac{e^{+k(z-2a)}}{2k}\,\beta\,\frac{(1-e^{-2\delta k})}{(1-\beta^2 e^{-2\delta k})}}_{\text{this is } \varphi_{ind}}\right]$$

Now the force is

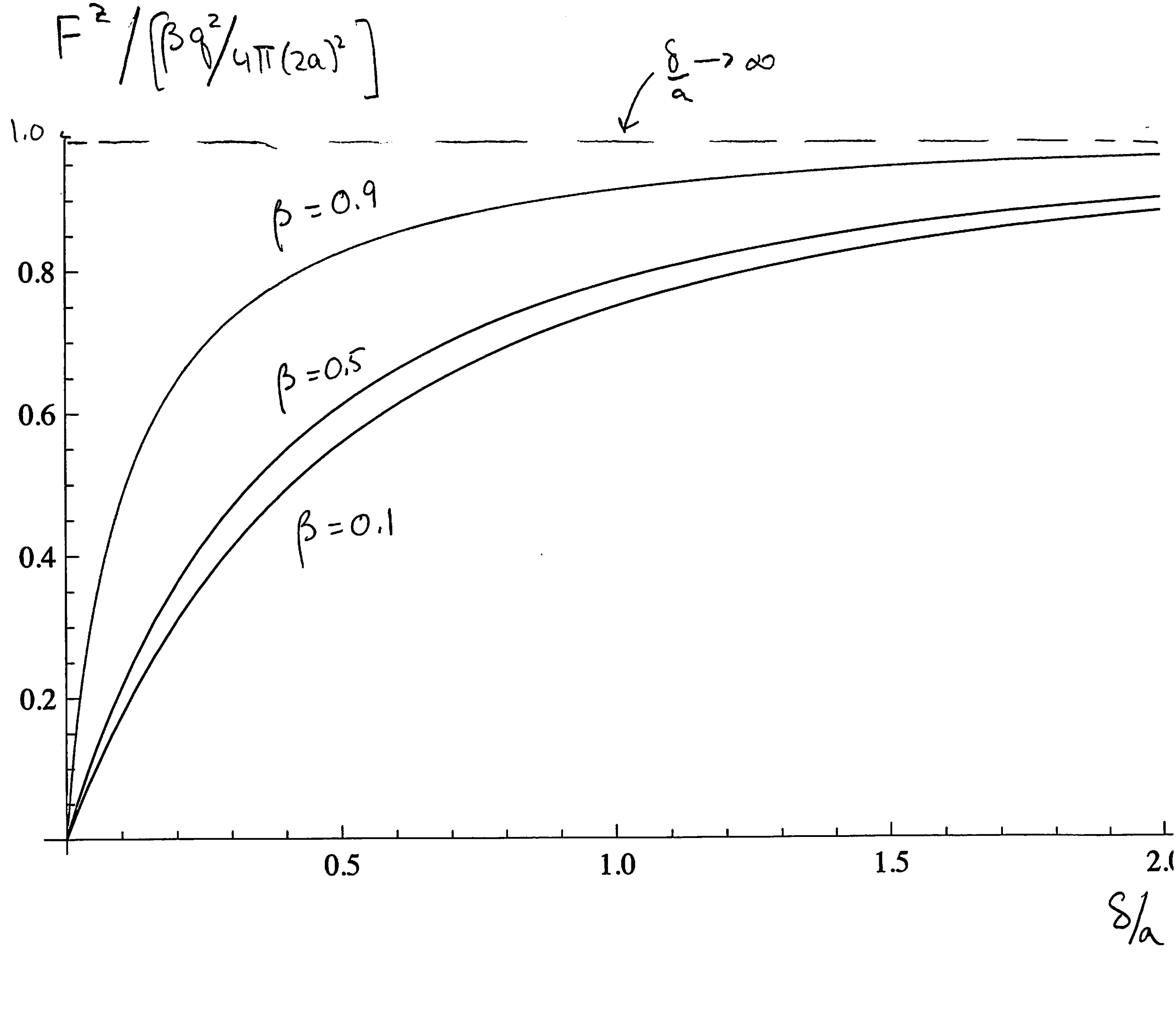
$$F = q\, E^z_{ind}$$

$$F = q\left(\frac{-\partial\varphi_{ind}}{\partial z}\right)\Bigg|_{z=0,\ r_\perp=0}$$

$$= q\left[q\int\frac{d^2k}{(2\pi)^2}\left[\frac{\cancel{k}}{2\cancel{k}_\perp}\, e^{-2ak}\,\beta\,\frac{(1-e^{-2\delta k})}{(1-\beta^2 e^{-2\delta k})}\right]\right]$$

$$F_z = \frac{q^2}{4\pi}\int_0^\infty k\,dk\; e^{-k2a}\,\frac{\beta\,(1-e^{-2\delta k})}{(1-\beta^2 e^{-2\delta k})}$$

$$F_z = \frac{\beta q^2}{4\pi a^2}\int_0^\infty u\,du\; e^{-2u}\,\frac{\beta\,(1-e^{-2(\delta/a)u})}{(1-\beta^2 e^{-2(\delta/a)u})}$$



$F^z / [\beta q^2 / 4\pi (2a)^2]$
$\frac{\delta}{a} \to \infty$
1.0
0.8
0.6
0.4
0.2
$\beta = 0.9$
$\beta = 0.5$
$\beta = 0.1$
0.5
1.0
1.5
2.0
δ / a