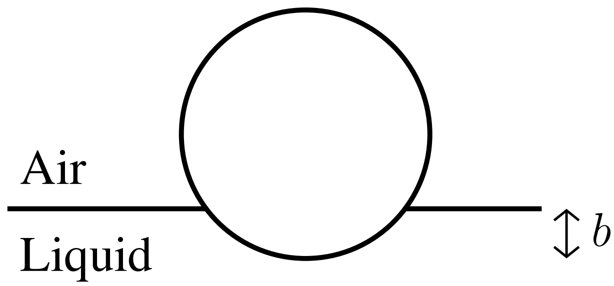
Problem 1. A half submerged metal sphere (UIC comprehensive exam)

A very light neutral hollow metal spherical shell of mass m and radius a is slightly submerged by a distance $b \ll a$ below the surface of a dielectric liquid. The liquid has mass density ρ and electrical permitivity ϵ. The liquid sits in air which has negligble density $\rho_o \ll \rho$, and the permitivity of air is approximately unity, $\epsilon_{\rm air} \simeq 1$. The pressure at the air liquid interface is p_0. Recall that stress tensor of an ideal fluid at rest is $T^{ij} = p(z)\delta^{ij}$ where $p(z)$ is the pressure as a function of z.



(a) Use the formalism of stress tensor to show that $p(z)$ increases as $p = p_0 + \rho g h$, where $h = -z$ is the depth below the surface, $z < 0$. Here p_0 is the pressure at the surface. Hint: what is the net force per volume for a static fluid?

(b) Use the formalism of stress tensor to prove that the boyancy force (for any shape) equals the difference in weight of the displaced fluid volume ΔV and the corresponding weight of the air:

$$F = (\rho - \rho_o) g \Delta V \simeq \rho g \Delta V.$$

Now a charge Q is added to the sphere, and the sphere becomes half submerged.

(c) Determine the potential, and the electrostatic fields E and D, in the top and lower halves of the sphere. Verify that all the appropriate boundary conditions are satisfied.

(d) What is the surface charge density on the top and lower halves of the sphere?

(e) Determine the electrostatic attractive force as a function of Q, a, and ϵ. What must Q be for the sphere to be half submerged? Make all reasonable approximations. Express your approximate result in terms of ρ, g, a, ϵ. Use dimensional reasoning to show that for a light sphere,

$$Q = \sqrt{\rho g a^5} \times \text{function of } \epsilon \, . \tag{1}$$

(f) (Optional) Estimate Q numerically for typical liquids.

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Problem: A half submerged sphere

a) The $\overset{\text{net}}{\vee}$ force $\overset{\text{per volume}}{\vee}$ on the fluid is zero:

$$-\partial_i T^{iz} - \rho g = 0 \qquad \text{(z-component of force)}$$

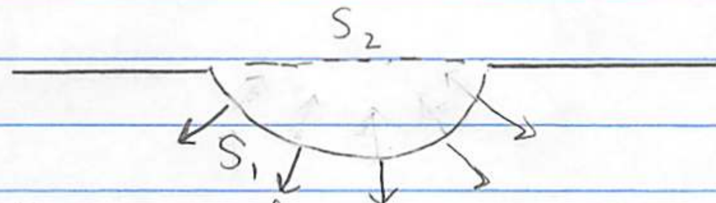
f^j from pressure $= -\partial_i T^{ij} = -\partial_j p$

gravity

$$-\partial_z p(z) = \rho g, \quad \text{so}$$

$$p(z) = p_0 - \rho g z$$

b)



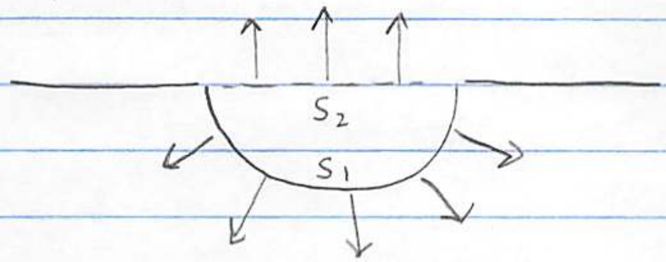
The Boyancy force is the difference in pressures on the shell in the water:

$$F_B^j = + \int_{S_1} da_i \left(T^{ij}_{air} - T^{ij}_{water} \right)$$

(see figure)

We are integrating over the surface S_1.
Then since the stress tensors are equal on the surface S_2,

we may integrate over the closed surface $S_1 + S_2$,



with outward directed normal as shown,

$$F_B^j = \oint_{S_1+S_2} da_i \left(T^{ij}_{air} - T^{ij}_{water}\right)$$

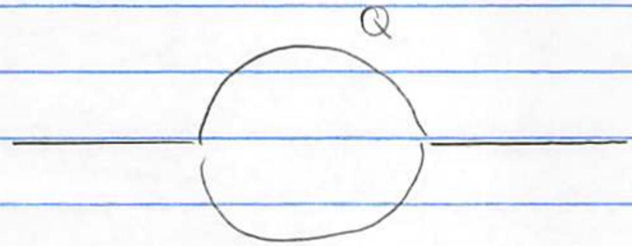
$$= \int_V dV \left[+ \partial_i T^{ij}_{air} - \partial_i T^{ij}_{water} \right]$$

$$F^j = \int dV \left[-\rho_{Air}\, g + \rho_W\, g \right] \delta^{jz}$$

$$F_B^j = \left(\rho_w - \rho_{air}\right) g V \,\delta^{jz} \simeq \rho_W\, g V \,\delta^{jz}$$

i.e. the boyancy force is upward, and its magnitude is equal to the weight of the displaced liquid.

d)



d) $\vec{E} = -\vec{\nabla}\Phi$ this is true in electrostatics even in media, where $\nabla \times \vec{E} = 0$.
For the potential take a form (everywhere)

$$\boxed{\Phi = \frac{C}{r}} \quad \text{and} \quad \boxed{\vec{E} = \frac{C}{r^2}\hat{r}}$$

The coefficient C is determined by the outgoing flux:

$$\int \vec{D}\cdot d\vec{a} = Q$$

Now

$$\boxed{\vec{D}_{upper} = \frac{C}{r^2}\hat{r}}, \quad \boxed{\vec{D}_{lower} = \varepsilon\frac{C}{r^2}\hat{r}}$$

So

$$\oint \vec{D}\cdot d\vec{a} = \frac{C}{a^2}(2\pi a^2) + \frac{\varepsilon C(2\pi a^2)}{a^2} = Q$$

Or

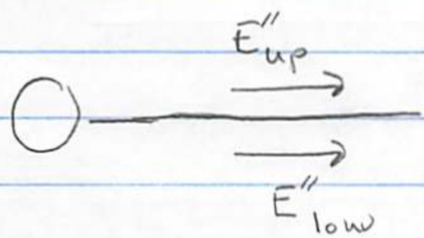
$$\boxed{C = \frac{Q}{4\pi}\frac{2}{\varepsilon+1}}$$

We can check that all B.C. are satisfied

These are:

① Φ = constant on surface of sphere ✓

② At the interface $E_{\parallel}$ is continuous:

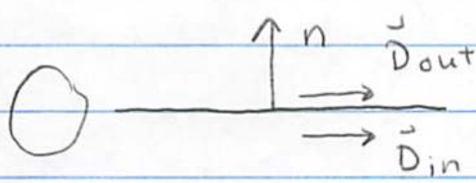


But, $E''_{low} = E''_{up} = \frac{C}{r^2}$

so $E_{\parallel}$ is clearly continuous.

③ The remaining condition is:

$$\vec{n} \cdot (\vec{D}_{out} - \vec{D}_{in}) = 0$$



But, from the symmetry of the problem

$$\vec{n} \cdot \vec{D}_{in} = \vec{n} \cdot \vec{D}_{out} = 0,$$

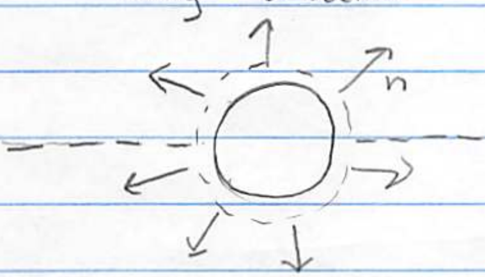
because the field lines are parallel to the surface. Thus the b.c. is also satisfied.

e) Then to compute the surface charge density we use

$$\sigma = n \cdot \vec{D} = \begin{cases} n \cdot \vec{E} & \text{upper half} \\ \vec{n} \cdot \vec{E}\, \varepsilon & \text{lower half} \end{cases}$$

$$= \begin{cases} \dfrac{Q}{2\pi a^2} \dfrac{1}{\varepsilon + 1} & \text{upper half} \\ \dfrac{Q}{2\pi a^2} \dfrac{\varepsilon}{\varepsilon+1} & \text{lower half} \end{cases}$$

f) To find the electrostatic force for charge Q we first recall that for any volume



$$F^j = -\oint da\, n_i T^{ij}$$

where we integrate over the surface indicated by the dashed lines, and

$$T^{ij} = -E^i D^j + \frac{1}{2} \vec{E} \cdot \vec{D}\, \delta^{ij}$$

where $\vec{D} = \vec{E}$ in the upper half and $\vec{D} = \varepsilon \vec{E}$ in the lower half

Let us compute F^z for the lower half

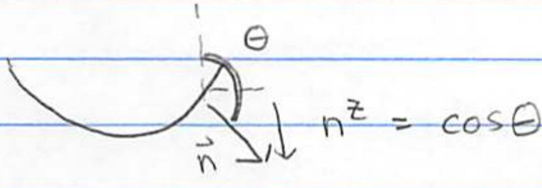
$$-n_i T^{ij} = \vec{n}\cdot\vec{E}\,(\varepsilon E^z) - \frac{1}{2}\varepsilon E^2 n^z$$

$$= E_0\,\varepsilon E_0 \cos\theta - \frac{1}{2}\varepsilon E_0^2 \cos\theta = \frac{1}{2}\varepsilon E_0^2 \cos\theta$$

Where we have defined:

$$E_0 \equiv \frac{Q}{4\pi a^2}\,\frac{2}{\varepsilon+1}$$ and used that

$$n_z = \cos\theta$$



Note $\cos\theta$ is less than zero, for $\theta > \pi/2$.

Now

$$F^z = a^2 \int_0^{2\pi} d\phi \int_{-1}^{0} d(\cos\theta)\; \frac{1}{2}\varepsilon E_0^2 \cos\theta$$

$$F^z_{low} = -\frac{2\pi}{4}\varepsilon E_0^2 a^2$$ ← this is the force in the lower half

Now the force from the upper half has the opposite sign, and does not have the leading factor of ε.

$$F^z_{up} = +\frac{2\pi}{4} E_0^2 a^2$$

So

$$F^z_{net} = -\frac{2\pi}{4} E_0^2 a^2 (\varepsilon - 1)$$

$$F^z = -\frac{Q^2}{32\pi a^2} \frac{4(\varepsilon-1)}{(\varepsilon+1)^2}$$

Now the charge Q should create enough of a downward pull to oppose the Boyance force. The dimensionfull parameters are

$$\rho_W \,, \; g \,, \; a \quad \text{while } \varepsilon \text{ is dimensonless}$$

And thus the charge which has dimensions $\sqrt{Nm^2}$ must be proportional to

$$(\rho g a^5)^{1/2}$$

Times a function of the dimensionless combos of parameters. Here we may use the three parameters ρ_W, g, a as our units of kg, m, s. Thus

$$Q = \sqrt{g \rho a^5} \; (\text{function of } \varepsilon)$$

Equating

$$\frac{Q^2}{32\pi a^2} \quad \frac{4(\varepsilon-1)}{(\varepsilon+1)^2} = \rho_w g \,\Delta V$$

with $\Delta V = \frac{1}{2} \times \frac{4}{3}\pi a^3$ gives

$$\boxed{Q = (\rho g a^5)^{1/2} \quad \frac{4\pi(\varepsilon+1)}{(3(\varepsilon-1))^{1/2}}}$$

g) For definiteness take $\rho = 1 g/cm^3$, $a = 1 cm$
$g = 10 \frac{m}{s^2} = 1000 \frac{cm}{s^2}$, $\varepsilon \simeq 2$

$$Q_{HL} = \frac{Q_{mks}}{\sqrt{\varepsilon_0}} \qquad \varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$$

$$= 8.85 \frac{(\mu C)^2}{Nm^2}$$

So

$$Q_{mks} = \mu C \left(\frac{8.85}{Nm^2} \rho g a^5 \right)^{1/2} \frac{4\pi(\varepsilon+1)}{(3(\varepsilon-1))^{1/2}}$$

$$\boxed{Q_{mks} = 64.7 \, nC \left(\frac{\rho}{1 g/cm^3} \right)^{1/2} \left(\frac{a}{1 cm} \right)^{5/2}}$$

this is a very modest amount of charge.

Problem 2. A cylinder in a magnetic field (Jackson)

A very long hollow cylinder of inner radius a and outer radius b of permeability μ is placed in an initially uniform magnetic field $\boldsymbol{B}_o$ at right angles to the field.

(a) For a constant field B_o in the x direction show that $A^z = B_o y$ is the vector potential. This should give you an idea of a convenient set of coordinates to use.

Remark: See Wikipedia for a list of the vector Laplacian in all coordinates. Most often the vector Laplacian is used if the current is azimuthal and solutions may be looked for with $A_\phi \neq 0$ and $A_r = A_\theta = 0$ (or $A_\rho = A_z = 0$ in cylindrical coordinates). This could be used for example in Problem 3. Similarly if the current runs up and down, with $A_z \neq 0$ and $A_\rho = A_\phi = 0$, so that $\boldsymbol{B} = (B_x(x,y,z), B_y(x,y,z), 0)$ is independent of z, then the vector Laplacian in cylindrical coordinates $-\nabla^2 A_z$ is a good way to go.

(b) Show that the magnetic field in the cylinder is constant $\rho < a$ and determine its magnitude.

(c) Sketch $|\boldsymbol{B}|/|\boldsymbol{B}_o|$ at the center of the as function of μ for $a^2/b^2 = 0.9, 0.5, 0.1$ for $\mu > 1$.

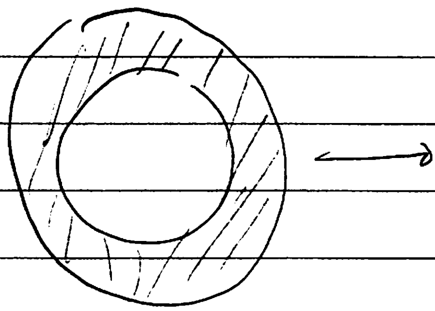
A - Cylinder in a magnetic field

(a) $$A^z = B_0 y$$

$$B_x = \frac{\partial A^z}{\partial y} - \overset{0}{\cancel{\frac{\partial A^y}{\partial z}}} \qquad \varepsilon^{lij}\partial_i A_j$$

$$B_x = B_0$$

(b)



$$-\nabla^2 A^z = 0$$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial A^z}{\partial \rho} + \frac{1}{\rho^2}\frac{-\partial^2}{\partial \phi^2}\right] A^z = 0$$

$$A_z = C_0 + D_0 \ln\rho + \sum_m \left(C_m \rho^m + \frac{D_m}{\rho^m}\right)\cos m\phi + \sum_m \left(E_m \rho^m + \frac{F_m}{\rho^m}\right)\sin m\phi$$

Given the B.C.

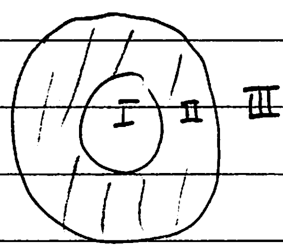
$$A^z \underset{\rho \to \infty}{\longrightarrow} B_0 y = B_0 \rho \sin\phi$$

We try a solution of the form:

$$A^z = E_1 \rho \sin\phi + \frac{F_1}{\rho} \sin\phi$$

In each region

$$A^z_{I} = E_{I} \rho \sin\phi$$



$$A^z_{II} = E_{II} \rho \sin\phi + \frac{F_{II}}{\rho} \sin\phi$$

$$A^z_{III} = B_0 \rho \sin\phi + \frac{F_{III}}{\rho} \sin\phi$$

At the interfaces we have

$$\left.\begin{array}{l} n \times (H_2 - H_1) = 0 \\ n \cdot (B_2 - B_1) = 0 \end{array}\right\}$$ 2 conditions + 2 interfaces = 4 constraints which is

sufficient to determine $E_{I}, E_{II}, F_{II}, F_{III}$

Given the B.C.

$$A^z \xrightarrow[\rho \to \infty]{} B_0 y = B_0 \rho \sin\phi$$

We try a solution of the form:

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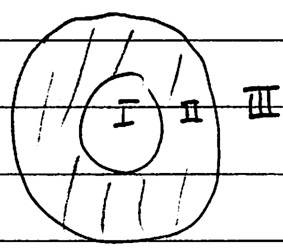

$$A^z_{II} = E_{II} \rho \sin\phi + \frac{F_{II}}{\rho} \sin\phi$$

$$A^z_{III} = B_0 \rho \sin\phi + \frac{F_{III}}{\rho} \sin\phi$$

At the interfaces we have

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$$\vec{B} = (\nabla \times A)$$

$$\vec{B} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\right) + \hat{\phi}\left(\frac{-\partial A_z}{\partial \rho}\right)$$

The continuity of B_ρ across interfaces: The B-conditions:

$$B_\rho^{I}\Big|_{\rho=a} = B_\rho^{II}\Big|_{\rho=b}$$ From the I, II interface

$$E_I a = E_{II} a + \frac{F_{II}}{a} \quad (1)$$

, and similarly at the II, III interface we have

$$E_{II} b + \frac{F_{II}}{b} = B_0 b + \frac{F_{III}}{b} \quad (2)$$

The H - conditions:

I-II: $$H_\phi^{II} = H_\phi^{I}$$ From the I & II interface

$$-\frac{1}{\mu}\frac{\partial A_z^{II}}{\partial \rho}\Big|_{\rho=a} = -\frac{1}{1}\frac{\partial A_z^{I}}{\partial \rho}\Big|_{\rho=a}$$

$$\frac{1}{\mu}\left(E_{II} - \frac{F_{II}}{a^2}\right) = E_I \quad (3)$$

And similarly

$$\frac{1}{\mu}\frac{\partial A_z^{II}}{\partial \rho}\bigg|_{\rho=b} = \frac{\partial A_z^{III}}{\partial \rho}\bigg|_{\rho=b}$$

$$\frac{1}{\mu}\left[E_{II} - \frac{F_{II}}{b^2}\right] = B_0 - \frac{F_{III}}{b^2} \qquad (4)$$

Solving (1), (2), (3), (4) setting $a \to 1$ so $b \to \frac{b}{a}$

$$E_I = E_{II} + F_{II}$$

$$E_I = \frac{1}{\mu}(E_{II} - F_{II})$$

$\Rightarrow$

$$\frac{(1+\mu)}{2} E_I = E_{II}$$

$$\frac{(1-\mu)}{2} E_I = F_{II}$$

Now

$$E_{II} + F_{II}/b^2 = B_0 + F_{III}/b^2$$

$$\left[(1+\mu) + \frac{(1-\mu)}{b^2}\right] E_I = 2B_0 + 2F_{III}/b^2$$

$$\left[(1+\mu) - \frac{(1-\mu)}{b^2}\right] E_I = 2\mu B_0 - \frac{2\mu}{b^2} F_{III}$$

So multiply the first eqn by μ and adding

$$\left[(\mu+1)^2 + \frac{(\mu-1)(1-\mu)}{b^2}\right] E_I = 4\mu B_0$$

And thus, restoring a, we find

$$E_I = \frac{4\mu}{(1+\mu)^2 - \frac{a^2}{b^2}(\mu-1)^2} B_0$$

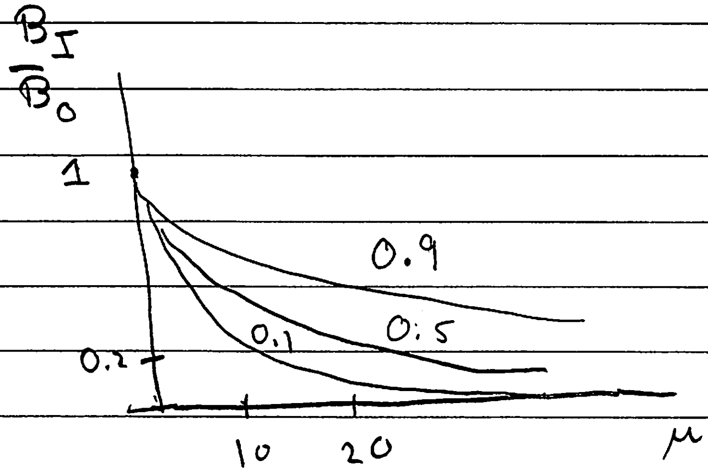
So plotting

$$B_I = (\nabla \times A^I) =$$

$$B_I^x = \frac{\partial A^{Iz}}{\partial y} = E_I$$

$$\boxed{B_I^x = \frac{4\mu}{(1+\mu)^2 - \frac{a^2}{b^2}(\mu-1)^2} B_0}$$

And



$B_I \simeq \frac{4}{(1-a^2)\mu}$ for large μ

Problem 3. Helmholtz coils (Jackson)

Consider a compact circular coil of radius a carrying current I, which lies in the $x-y$ plane with its center at the origin.

(a) By elementary means compute the magnetic field along the z axis.

(b) Show by direct analysis of the Maxwell equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = 0$ that slightly off axis near $z = 0$ the magnetic field takes the form

$$B_z \simeq \sigma_0 + \sigma_2 \left(z^2 - \tfrac{1}{2}\rho^2\right), \quad B_\rho \simeq -\sigma_2 z \rho\,, \tag{2}$$

where $\sigma_0 = (B_z^o)$ and $\sigma_2 = \frac{1}{2}\left(\frac{\partial^2 B_z^o}{\partial z^2}\right)$ are the field and its z derivatives evaluated at the origin. For later use give σ_0 and σ_2 explictly in terms of the current and the radius of the loop.

Remark: The magnetic field near the origin satisfies $\nabla \times \boldsymbol{B} = 0$, so $\nabla \cdot \boldsymbol{B} = 0$. We say it is harmonic function[1]. Because the function is harmonic, the taylor series of B on the z axis, is sufficient to determine the taylor series close to the z axis.

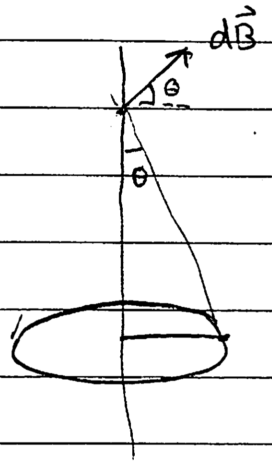
(c) Now consider a second identical coil (co-axial with the first), having the same magnitude and direction of the current, at a height b above the first coil, where a is the radii of the rings. With the coordinate origin relocated at the point midway between the two centers of the coils, determine the magnetic field on the z-axis near the origin as an expansion in powers of z to z^4. Use mathematica if you like. You should find that the coefficient of z^2 vanishes when $b = a$

Remark For $b = a$ the coils are known as Helmholtz coils. For this choice of b the z^2 terms in part (c) are absent. (Also if the off-axis fields are computed along the lines of part (b), they also vanish.) The field near the origin is then constant to 0.1% for $z < 0.17\,a$.

[1] This means that $\boldsymbol{B}$ can be written $\boldsymbol{B} = -\nabla\psi$ where $-\nabla^2\psi = 0$

Problem 2 - HelmHoltz coils

a)



$$d\vec{B} = \frac{I}{c}\frac{d\vec{l} \times \hat{r}}{4\pi r^2}$$

$$dB_z = \frac{I}{c}\frac{dl \sin\theta}{4\pi(z^2+a^2)}$$

So

$$B_z = \frac{I}{c}\frac{2\pi a^2/4\pi}{(z^2+a^2)^{3/2}} = \boxed{\frac{I}{c}\frac{(\pi a^2)}{4\pi}\frac{2}{(z^2+a^2)^{3/2}}}$$

b) Now using

$$\nabla \cdot B = 0 = \partial_z B^z + \frac{1}{\rho}\partial_\rho(\rho B^\rho) = 0$$

For ρ small: $B^\rho = \rho\, C(z)$

$$\partial_z B^z = -\frac{1}{\rho}\partial_\rho(\rho\, \rho C(z))$$

$$\partial_z B^z = -2C(z)$$

For z small

$$B(z) \simeq B_0 + \frac{1}{2}B'' z^2 \equiv \sigma_0 + \sigma_2 z^2$$

Where

$$B(z) \simeq \underbrace{\frac{I}{2ac}}_{B_0} \underbrace{- \frac{3}{4} \frac{I}{ca^3} z^2}_{+\frac{1}{2}B''}$$

So for the current case

$$\partial_z B^z \simeq 2\sigma_2 z$$

And

$$B^\rho \simeq -\frac{1}{2} \partial_z B^z \rho$$

$$\boxed{B^\rho \simeq -\sigma_2 z \rho} \quad \text{ⓦ} \quad \boxed{\sigma_2 = -\frac{3}{4} \frac{I}{ca^3}}$$

So then from:

$\nabla \times B = 0$ we have

$$\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} = 0, \quad \text{or integrating:}$$

$B_z(0)$ means $B_z(\rho=0, z)$, i.e. $\rho = 0$ but finite z.

$$\int_0^\rho \frac{\partial B_\rho}{\partial z} d\rho' = B_z(\rho) - B_z(0)$$

$$\int_0^\rho -\sigma_2 \rho' d\rho' = B_z(\rho) - B_z(0)$$

$$B_z(0) + -\frac{1}{2}\sigma_2 \rho^2 = B_z(\rho)$$

So using

$$B_z(0) = \sigma_0 + \sigma_2 z^2$$

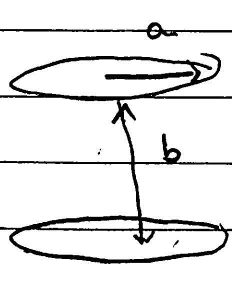
So

$$\boxed{B_z(\rho) \approx \sigma_0 + \sigma_2\left(z^2 - \frac{1}{2}\rho^2\right)}$$

Where $\boxed{\sigma_0 = \frac{I}{2ac} \quad \text{and} \quad \sigma_2 = -\frac{3}{4}\frac{I}{ca^3}}$

c) Using the formulas for B of a ring

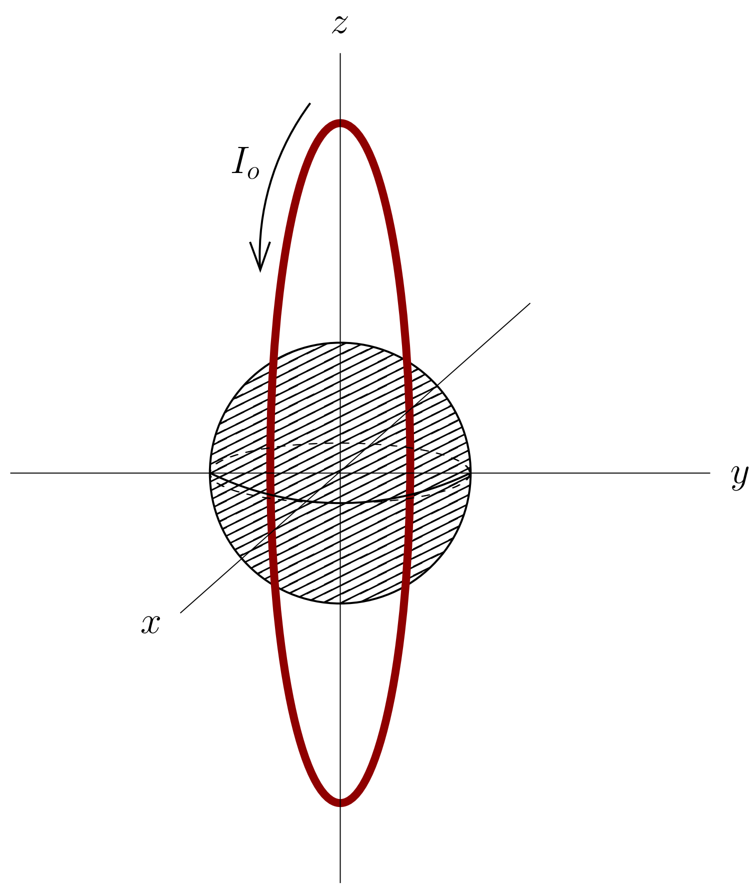
$$B_z = \frac{I}{2c}\left[\frac{a^2}{((z+b/2)^2 + a^2)^{1/2}} + \frac{a^2}{((z-b/2)^2 + a^2)^{1/2}}\right]$$



$$\simeq \frac{I}{2c}\left[\frac{2a^2}{(a^2+(b/2)^2)^{3/2}} - \frac{3\,a^2(a^2-b^2)}{(a^2+(b/2)^2)^{7/2}}\,z^2 + \frac{15}{8}\,a^2\frac{(2a^4 - 6a^2b^2 + b^4)}{(a^2+(b/2)^2)^{11/2}}\,z^4\right]$$

Problem 4. A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



(a) Determine the bound surface current on the surface of the sphere.

(b) Write down (no long derivations please) the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere by analogy with the spinning charged sphere disucssed in class.

(c) Show that your solution satisfies the boundary conditions of magnetostatics on the surface of the sphere.

(d) Compute the net-torque on the circular hoop. Indicate the direction and interpret.

Solution

(a) The surface current is given by the discontinuity in the magnetization

$$\frac{\boldsymbol{K}_b}{c} = \boldsymbol{n} \times (\boldsymbol{M}_{\rm out} - \boldsymbol{M}_{\rm in}) \tag{8}$$

Then since $\boldsymbol{M}_{\rm out} = 0$ we have

$$\frac{\boldsymbol{K}_b}{c} = -M(\boldsymbol{n} \times \hat{\boldsymbol{z}}) \tag{9}$$

where $M = m/(\frac{4}{3}\pi a^3)$ is the magnetic dipole moment per volume. With $\boldsymbol{n} = \hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{z}} = -\sin\theta\hat{\boldsymbol{\theta}} + \cos\theta\hat{\boldsymbol{r}}$, we find

$$\frac{\boldsymbol{K}_b}{c} = \frac{3}{4\pi a^3} m \sin\theta\, \hat{\boldsymbol{\phi}} \tag{10}$$

(b) A formal analogy with the charged spinning sphere gives the solution. In the spinning sphere case the surface current was also $\propto \sin\theta\,\hat{\boldsymbol{\phi}}$. In the spinning sphere case we found that the magnetic field outside is one of a magnetic dipole, where all of magnetic moment is placed at the origin

$$\boldsymbol{B} = \frac{1}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \mathbf{m}\right] \tag{11}$$

Or more explicitly

$$\boldsymbol{B} = \frac{1}{4\pi r^3} 2m\cos\theta\,\hat{\boldsymbol{r}} + \frac{1}{4\pi r^3} m \sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{12}$$

Inside sphere, the magnetic field was constant

$$\boldsymbol{B} = B_o\,\hat{\boldsymbol{z}} \tag{13}$$

The constant B_o can be picked off from the boundary conditions as we will do in the next item.

(c) The boundary conditions read

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0 \tag{14}$$
$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{15}$$

Then from the second boundary condition at $r = a$

$$B_r|_{\rm out} = B_r|_{\rm in}\ . \tag{16}$$

With the magnetic field outside the sphere

$$B_r|_{\rm out} = \frac{1}{4\pi r^3} 2m\cos\theta\,, \tag{17}$$

and inside the sphere

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{B}|_{\rm in} = B_o\, \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} = B_o \cos\theta \,, \tag{18}$$

comparison at $r = a$ gives

$$B_o = \frac{1}{4\pi a^3} 2m \,. \tag{19}$$

Thus we have shown that $\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0$ is satisfied, provided B_o takes a specific value.

Note that inside the sphere with $M = m/(4\pi a^3/3)$ and the inductance is

$$H_{z,in} = H_o = B_o - M = -\frac{m}{4\pi a^3} \,. \tag{20}$$

Now one can verify the parallel boundary condition $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0$ using eq. (20)

$$H_{\theta,\rm out} - H_{\theta,\rm in}|_{r=a} = \left. \left(\frac{1}{4\pi r^3} m \sin\theta + H_o \sin\theta \right) \right|_{r=a} = 0 \tag{21}$$

(d) To compute the torque we first compute the lorentz force on a element of length $d\ell = b d\theta$.

$$dF = \frac{I_o}{c} d\ell B_\perp \tag{22}$$

$$= \frac{I_o}{c} b d\theta \; B_r \tag{23}$$

$$= \frac{I_o}{c} b d\theta \; \frac{2m\cos\theta}{4\pi b^3} \tag{24}$$

The right hand rule indicates that the force is in the $-\hat{\boldsymbol{y}}$ direction in the upper hemisphere, and in the positive $\hat{\boldsymbol{y}}$ direction in the lower hemisphere. This implies that the net torque points along the x-axis. This can be intuited by noting that the magnetic moment of the hoop tends to align with the magnetic field from the sphere

(e) The torque around the x-axis

$$\tau = \int d\tau = \int b \cos\theta \, dF \tag{25}$$

$$= 2 \int_0^\pi b\cos\theta \; \frac{I_o}{c} b d\theta \; \frac{2m\cos\theta}{4\pi b^3} \tag{26}$$

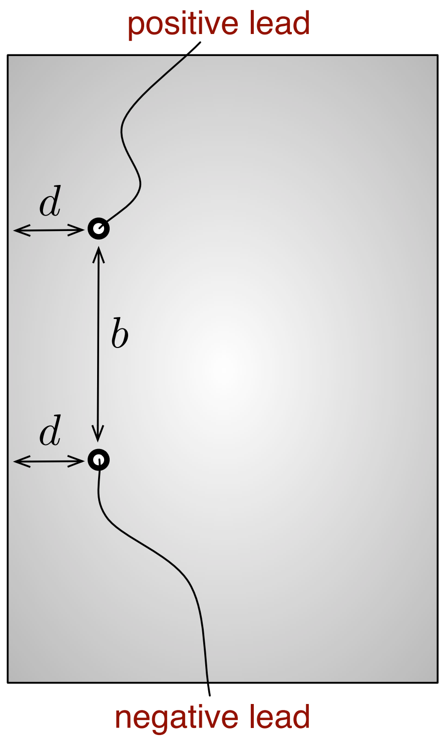
$$= \frac{4m(I_o/c)b^2}{4\pi b^3} \int_0^\pi d\theta \cos^2\theta \tag{27}$$

$$= \frac{4m(I_o/c)b^2}{4\pi b^3} \frac{\pi}{2} \tag{28}$$

$$= \frac{2m}{4\pi b^3} \left[\frac{I_o}{c} \pi b^2 \right] \tag{29}$$

Problem 5. Electrodes in an ohmic material filling half of space

Two small spherical electrodes of radius a are embedded in a semi-infinite medium of conductivity σ, each at a distance $d \gg a$ from the plane face of the medium and at a distance $b \gg a$ from each other.



(a) State the boundary conditions on all surfaces. Assume that the electrodes emit and absorb a total current I which is spread uniformly over the surface of the sphere. Do not assume that a is small for this part (but of course $a < b$ and $a < d$).

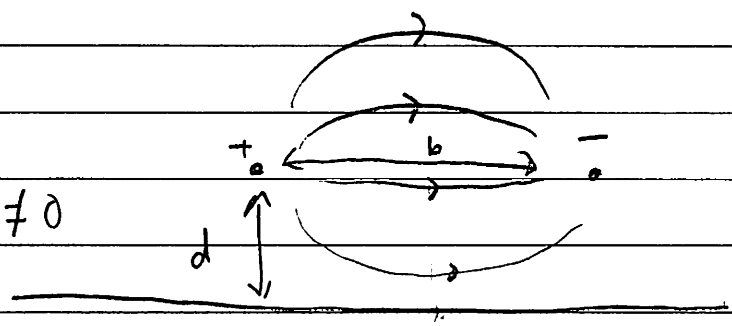
(b) Argue for small a the potential between the two electrodes satisfies

$$\nabla^2 \varphi = \frac{I}{\sigma}\delta^3(\boldsymbol{r} - \boldsymbol{r}_1) - \frac{I}{\sigma}\delta^3(\boldsymbol{r} - \boldsymbol{r}_2) \tag{3}$$

where $\boldsymbol{r}_1$ is the position of the emitting electrode while $\boldsymbol{r}_2$ is the position of the absorbing electrode.

(c) Find the resistance between the electrodes. Sketch the flow lines of current if the two electrodes are held at a potential difference ΔV. (Hint: use images to solve Eq. (3) with the right boundary conditions.)

Problem 4 Two electrodes in a conductor filling half space



$\sigma \neq 0$

$\sigma = 0$

We want to solve;

$$\vec{\nabla}\cdot(\overbrace{\sigma\vec{E}}^{\vec{j}}) = I\left[\delta^3(\vec{r}_1) - \delta^3(\vec{r}_2)\right]$$

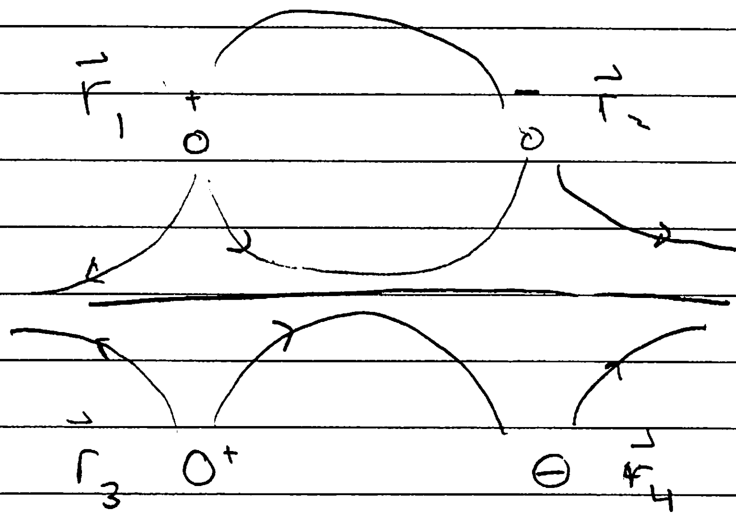
$$\vec{r}_1 = (x_1, y_1, z_1) = \left(-\frac{b}{2}, 0, d\right)$$

$$r_2 = (x_2, y_2, z_3) = \left(\frac{b}{2}, 0, d\right)$$

Here C is related to the total current

$$\int_{\substack{\text{Small}\\ \text{spher}}} d^3r\ \nabla\cdot j = \int da\cdot\vec{j} = I$$

So we place images:



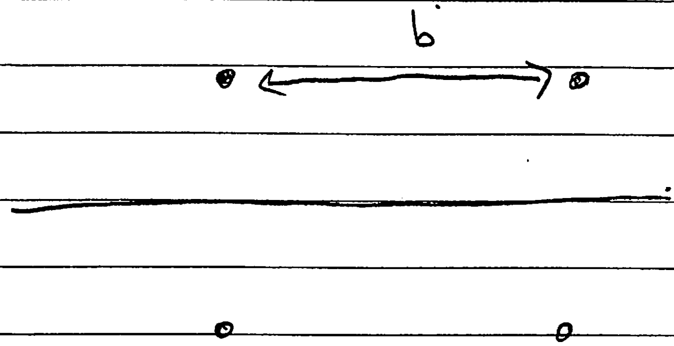
So,

$$\varphi = \frac{+I/\sigma}{4\pi|\vec{r}_1 - \vec{r}_1|} - \frac{I/\sigma}{4\pi|\vec{r} - \vec{r}_2|}$$

$$+ \frac{I/\sigma}{4\pi|\vec{r} - \vec{r}_3|} - \frac{I/\sigma}{4\pi|\vec{r} - \vec{r}_4|}$$

So to find the potential difference we want

$$\Delta\varphi = \varphi\Bigg|_{\substack{r \to r_1 \\ |\vec{r} - r_1| = a}} - \varphi\Bigg|_{\substack{r \to r_2 \\ |\vec{r} - \vec{r}_2| = a}}$$



$$\varphi\Big|_{|\vec{r}-r_1|\simeq a} = \frac{I/\sigma}{4\pi a} - \frac{I/\sigma}{4\pi b} + \frac{I/\sigma}{4\pi(2d)} - \frac{I/\sigma}{4\pi\left((2d)^2+b^2\right)^{1/2}}$$

$$\varphi\Big|_{|r-r_2|\simeq a} = \frac{+I/\sigma}{4\pi(b)} - \frac{I/\sigma}{4\pi a} + \frac{I/\sigma}{4\pi\sqrt{(2d)^2+b^2}} - \frac{I/\sigma}{4\pi(2d)}$$

= Same but opposite sign

So

$$\boxed{\Delta\varphi = I \times \underbrace{\frac{1}{2\pi\sigma}\left[\frac{1}{a} - \frac{1}{b} + \frac{1}{2d} + \frac{1}{\sqrt{(2d)^2+b^2}}\right]}_{\text{Resistance}}}$$