

### Problem 1. Energy of a wire and rectangle (Jackson)

- (a) Consider an infinitely long straight wire carrying a current  $I$  in the  $z$  direction. Use the known magnetic field of this wire, and the integral form of  $\mathbf{B} = \nabla \times \mathbf{A}$

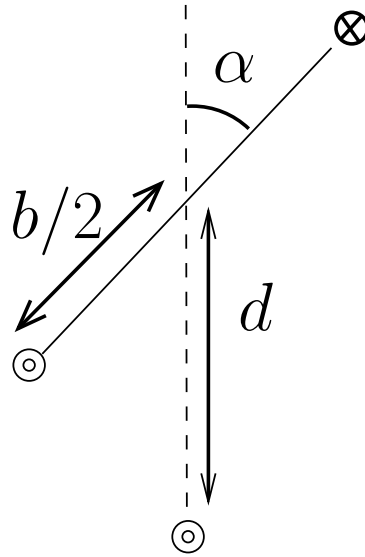
$$\int_S \mathbf{B} \cdot d\mathbf{S} = \oint d\boldsymbol{\ell} \cdot \mathbf{A} \quad (1)$$

to show that the vector potential for an infinite current carrying wire in the Coulomb gauge is

$$A^z = \frac{-(I/c)}{2\pi} \log \rho + \text{const} \quad (2)$$

Check that the Coulomb gauge condition is satisfied.

- (b) Now consider a flat right rectangular loop carrying a constant current  $I_1$  that is placed near a long straight wire carrying a constant current  $I_2$ . The rectangular loop is oriented so that its center is a perpendicular distance  $d$  from the wire; the sides of length  $a$  are parallel to the wire and the sides of length  $b$  make an angle  $\alpha$  with the plane containing the wire and the loops center (the dashed line below). In the schematic diagram below, the current  $I_2$  in the long wire flows out of page. The orientation of  $I_1$  is also indicated, i.e. the current lower edge of the rectangle (of length  $a$ ) also comes out of the page.



Show that the interaction energy

$$W_{12} = \frac{I_1}{c} F_1 \quad (3)$$

(where  $F_1$  is the magnetic flux from  $I_2$  through the rectangular circuit carrying  $I_1$ ), is

$$W_{12} = \frac{aI_1I_2}{4\pi c^2} \ln \left[ \frac{4d^2 + b^2 + 4db \cos \alpha}{4d^2 + b^2 - 4db \cos \alpha} \right] \quad (4)$$

- (c) (Optional) Using energy considerations calculate the force between the loop and the wire for constant currents. (This requires some somewhat lengthy differentiation and is best done with mathematica) . You should find

$$F^x = \frac{I_1 I_2 a}{4\pi c^2} \left( \frac{8b(b^2 - (2d)^2) \cos \alpha}{b^4 + (2d)^4 - 2b^2(2d)^2 \cos(2\alpha)} \right), \quad (5)$$

$$F^y = \frac{I_1 I_2 a}{4\pi c^2} \frac{1}{d} \left( \frac{4b(2d)(b^2 + (2d)^2) \sin \alpha}{b^4 + (2d)^4 - 2b^2(2d)^2 \cos(2\alpha)} \right), \quad (6)$$

where in the figure above the  $x$  and  $y$  axes are respectively perpendicular and parallel to the separation  $d$ . For large  $d \gg a, b$  the force expands to

$$F^y = - \frac{I_1 I_2 ab \cos \alpha}{2\pi c^2 d^2} \quad (7a)$$

$$F^x = + \frac{I_1 I_2 ab \sin \alpha}{2\pi c^2 d^2} \quad (7b)$$

- (d) (Optional) Check that for large distances  $d \gg a, b$  the force computed in the previous sub-question agrees with the appropriate formula for a dipole in an external field.
- (e) Show that when  $d \gg a, b$  the interaction energy reduces to  $W_{12} = \mathbf{m} \cdot \mathbf{B}$ , where  $\mathbf{m}$  is the magnetic moment of the loop. Explain the sign.

## Problem 2. Dipole two ways

Consider two charges  $\pm q$  moving a short distance on  $\ell$  the  $z$  axis forming a dipole, i.e.

$$z_+(t) = \ell/2 \cos(\omega t) \quad (8)$$

$$z_-(t) = -\ell/2 \cos(\omega t) \quad (9)$$

This charged has dipole moment  $p = q\ell \cos(\omega t)$  directed in the  $z$  direction

- (a) Write down the electric field as a function of time.
- (b) By treating the time dependent electric field as a “displacement current”

$$\frac{\mathbf{j}_D}{c} = \frac{1}{c} \partial_t \mathbf{E}, \quad (10)$$

determine the magnetic field using Ampere’s Law

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \int_S d\mathbf{a} \cdot \frac{\mathbf{j}_D}{c} \quad (11)$$

with an appropriate Amperian loop. Assume that  $\mathbf{B} = B(r, \theta) \hat{\phi}$  which can be justified from the symmetry of the problem. You should find

$$\mathbf{B} = \frac{\dot{\mathbf{p}} \times \mathbf{n}}{4\pi r^2 c} \quad (12)$$

- (c) Consider the Biot Savat Law

$$\mathbf{B}(\mathbf{r}) = \int d^3\mathbf{r}_0 \frac{\mathbf{j}(\mathbf{r}_0)}{c} \times \frac{(\mathbf{r} - \mathbf{r}_0)}{4\pi |\mathbf{r} - \mathbf{r}_0|^3} \quad (13)$$

Show that if  $\mathbf{j}$  is curl free  $\nabla \times \mathbf{j} = 0$  the magnetic field is zero. Explain why the displacement current from an electrostatic field does not need to be included in the Biot-Savat Law.

Thus we see that the *electrostatic* displacement current is necessary for consistency of Ampere’s Law but does not actually produce a magnetic field.

- (d) Show (using a high school physics argument) that the current density integrated over small volume  $\Delta V$  surrounding the charges is

$$\mathbf{j} \Delta V = \partial_t \mathbf{p} \quad (14)$$

Use the Biot-Savat Law to determine the magnetic field. It should agree with (b).

- (e) At what radius do the electric fields part (a) and the magnetic fields of part (b), (d) become equal in magnitude. At this radius the approximation scheme is no longer valid.

### Problem 3. A rotating magnet

A magnetic dipole moment of magnitude  $m$  lying in the  $xy$  plane rotates about its center with angular velocity  $\omega$ . It points in the  $x$  direction at time  $t = 0$

- (a) Find the electric and magnetic fields on the  $z$  axis as a function of time. Work to the lowest non-trivial order in inverse powers of  $c$ . (Hint: the easy way is to use the vector potential to determine  $\mathbf{E}$ )
- (b) Estimate the magnitude of  $E/B$  at a given radius  $r$ . At what radius is the solution you found in part (a) valid? At what radius does it break down and why?

## Problem 4. Electric and Magnetic fields of AC Solenoid

A cylindrical solenoid of high conductivity and radius  $a$  carries surface current  $\mathbf{K} = K_o \cos(\omega t) \hat{\phi}$

- (a) Determine the electric and magnetic fields to the first non-vanishing order in the quasi-static approximation.
- (b) Show that the magnetic field to the next-to-leading order in the quasi-static approximation outside the cylinder is

$$\Delta B = \delta B_z(\rho) - \delta B_z(\rho_{\max}) = \frac{K_o}{c} \cos(\omega t) \frac{1}{2} (\omega a/c)^2 \left( -\log \frac{\rho}{a} + C \right) \quad (15)$$

where  $C = \log \rho_{\max}/a$ . Here we are quoting  $\Delta B$  the difference between  $\delta B$  at  $\rho$  and  $\delta B$  at  $\rho_{\max}$ .

- (c) The cutoff  $\rho_{\max}$  arises because the quasi static approximation breaks down for large  $\rho$  where the physics of radiation becomes important.  $\rho_{\max}$  should be of order  $\rho_{\max} \sim c/\omega$ . Explain qualitatively why the approximation breaks down for this radius.

**Remark:** Certainly  $|\delta B_z(\rho_{\max})|$  is logarithmically smaller than  $|\delta B_z(\rho)|$  for  $\rho \sim a$ . In a logarithmic approximation we can neglect  $\delta B_z(\rho_{\max})$  and set  $\rho_{\max} = c/\omega$  leading to

$$\delta B_z(\rho) \simeq \frac{K_o}{c} \cos(\omega t) \frac{1}{2} (\omega a/c)^2 \left( -\log \frac{\rho}{a} + \log(c/(\omega a)) \right) \quad (16)$$

Leading log accuracy may not be familiar to you. It just says that we are neglecting the constant inside the logarithm which is of order 1. Thus in this approximation,

$$\log(100/2) = \log(100) - \log(2) \simeq \log(100) \quad (17)$$

$$3.9 \simeq 4.6 \quad (18)$$

which is often good enough for government work. Bethe famously used such approximations to estimate the first QED corrections to the hydrogen spectrum.

- (d) Determine the magnetic field (to the next-to-leading order in the quasi-static approximation) inside the cylinder to logarithmic accuracy, and qualitatively sketch the complete magnetic field  $B(\rho)/B_o$  where  $B_o$  is the leading order answer in the center of the cylinder.

**Remark:** Note that the  $\rho$  dependence of part (b) and part (c) does not depend on the value of  $C = \log(\rho_{\max}/a)$ .

- (e) Determine the vector and scalar potentials in the Coulomb and Lorentz gauges to the required order and accuracy to reproduce the electric and magnetic fields in part (a) and verify that you obtain the correct fields.

### Problem 5. Dipole down the tube (Zangwill)

A small magnet (weight  $w$ ) falls under gravity down the center of an infinitely long, vertical, conducting tube of radius  $a$ , wall thickness  $D \ll a$ , and conductivity  $\sigma$ . Let the tube be concentric with the  $z$ -axis and model the magnet as a pointlike dipole with moment  $\mathbf{m} = m\hat{\mathbf{z}}$ . We can find the terminal velocity of the magnet by balancing its weight against the magnetic drag force associated with the ohmic loss in the walls of the tube.

- (a) At the moment it passes through  $z = z_o(t)$ , show that the magnetic flux produced by  $\mathbf{m}$  through the a ring of radius  $a$  at height  $z'$  is

$$\Phi_B = \frac{m a^2}{2 r_o^3} \quad \text{where} \quad r_o^2 = a^2 + (z_o - z')^2 \quad (19)$$

- (b) When the speed  $v$  of the dipole is small, argue that the Faraday EMF induced in the ring is

$$\mathcal{E} = -\frac{1}{c} \partial_t \Phi_B = \frac{v}{c} \frac{\partial \Phi_B}{\partial z'} \quad (20)$$

- (c) Show that the current induced in the thin slice of tube which includes the ring is

$$dI = \frac{3mav\sigma D}{4\pi c} \frac{(z_o - z')}{r_o^5} dz' \quad (21)$$

- (d) Compute the magnetic drag force  $\mathbf{F}$  on  $\mathbf{m}$  by equating the rate at which the force does work to the power dissipated in the walls of the tube by Joule heating. I find

$$\frac{dE_{\text{ohm}}}{dt} = \sigma D (v/c)^2 \frac{m^2}{a^4} \left( \frac{45}{1024} \right) \quad (22)$$

- (e) Find the terminal velocity of the magnet.

## Problem 6. Eddy-Current Levitation (Zangwill)

A wire loop of radius  $b$  in the  $x - y$  plane carries a time-harmonic current  $I_o \cos \omega t$ . Find the value of  $I_o$  needed to levitate a small sphere of mass  $m$ , radius  $a$ , and conductivity  $\sigma$  at a height  $z$  above the center of the loop. Assume  $a \ll b$  and that  $\delta \ll a$  where  $\delta$  is the skin depth of the sphere.

- (a) First recall in class that we showed that an oscillating magnetic field is damped out over a distance  $\delta$ . So the picture is that a surface current is generated to satisfy the boundary conditions:

$$\mathbf{n} \times (\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}) = \mathbf{K} \quad (23)$$

or

$$\mathbf{n} \times (\mathbf{H}_{\text{out}}) = \mathbf{K} \quad (24)$$

since  $\mathbf{H}_{\text{in}} = 0$ . In reality the “surface” current has a thin thickness of order  $\delta$ , and integrating the current  $\mathbf{j}$  over the thickness of order  $\delta$  gives  $\mathbf{K}$  (as you did in the inclass exercise).

The boundary conditions satisfied by the magnetic field at the surface of the sphere are therefore

$$\mathbf{n} \cdot \mathbf{B}_{\text{out}} = 0 \quad (25)$$

and

$$\mathbf{n} \times \mathbf{B}_{\text{out}} = \mathbf{K} \quad (26)$$

Solve for the magnetic fields in the vicinity of the sphere (which are affected by the surface currents) with the boundary conditions given above, and the requirement that the field should asymptote to the field of the ring far from the sphere in units of the sphere radius,  $a$ . We are, however, still talking about distances very close to the sphere in units the ring radius  $b$ , i.e. for

$$a \ll r \ll b \quad (27)$$

the magnetic field approaches the magnetic field of the ring without the sphere at height  $z$ .

As an intermediate step show that the vector potential outside the sphere is

$$A_\phi(r, \theta) = \frac{1}{2} B_o r \sin \theta - \frac{B_o a^3}{2r^2} \sin(\theta) \quad (28)$$

using the separation of variables technique. Sketch the lines of magnetic field.

- (b) This solution to the fields in the vicinity of the sphere will determine the surface current  $\mathbf{K}$  and hence the induced magnetic moment on the sphere. Show that

$$\mathbf{m} = -\hat{\mathbf{z}} (2\pi a^3) B_o(z) \quad (29)$$

where  $B_o(z)$  is the field from the ring at height  $z$ . Note that the magnetic moment opposes the applied field.

- (c) Then you can compute the force by using the dipole moment and familiar formulas for forces on dipoles in external fields. I find that the time averaged force is

$$\overline{F}_z = \frac{3\pi}{4} \left( \frac{I_o}{c} \right)^2 \frac{a^3 b^4 z}{(z^2 + b^2)^4} = mg \quad (30)$$