Problem 1. (Optional) Electric field in the far field

If you get stuck check the notes online. The scalar and vector potential in the far field are

$$\varphi(t, \boldsymbol{r}) = \frac{1}{4\pi r} \int d^3\boldsymbol{r}_o \rho(T, \boldsymbol{r}_o) \tag{1}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{1}{4\pi r} \int d^3\boldsymbol{r}_o \boldsymbol{J}(T, \boldsymbol{r}_o)/c \tag{2}$$

where the retarded time $T = t - |\boldsymbol{r} - \boldsymbol{r}_o|/c$ in the far field is

$$T = t - r/c + \frac{\boldsymbol{n} \cdot \boldsymbol{r}_o}{c} \tag{3}$$

The goal is to compute the electric field

$$\boldsymbol{E}(t, r) = -\frac{1}{c}\partial_t \boldsymbol{A}(t, \boldsymbol{r}) - \nabla\varphi(t, \boldsymbol{r}) \tag{4}$$

(a) (Optional) Consider the change of variable $t, \boldsymbol{r}_o \to T, \boldsymbol{r}_o$. Show that

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} \tag{5}$$

$$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t - \frac{\boldsymbol{n}}{c}\frac{\partial}{\partial t} \tag{6}$$

(b) (Optional) Compute

$$\left(\frac{\partial}{\partial t} + c\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) T \tag{7}$$

You should find a simple result. Interpret the answer using the definition of T

$T \equiv$ the time when the light should be emitted from $\boldsymbol{r}_o$ to arrive at the observation point $(t, \boldsymbol{r})$.

How do you interpert the derivative:

$$\left(\frac{\partial}{\partial t} + c\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) \tag{8}$$

(c) (Optional) Show that

$$\boldsymbol{E} = -\frac{1}{4\pi r c^2} \int_{r_o} \frac{\partial \boldsymbol{J}(T, r_o)}{\partial T} + \frac{\boldsymbol{n}}{c}\frac{1}{4\pi r} \int_{r_o} \frac{\partial \rho(T, r_o)}{\partial T} \tag{9}$$

(d) (Optional) Use current conservation to express

$$\frac{\partial \rho(T, \boldsymbol{r}_o)}{\partial T} = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_T = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_t + \frac{\boldsymbol{n}}{c} \cdot \frac{\partial \boldsymbol{J}}{\partial T} \tag{10}$$

where $(\nabla_{r_o} \cdot \boldsymbol{J})_t$ denotes the divergence at fixed observation time

(e) (Optional) Conclude that only the transverse piece of the current to $\boldsymbol{n}$ contributes to the radiation field

$$\boldsymbol{E} = -\frac{1}{4\pi r}\frac{1}{c^2}\int_{\boldsymbol{r}_o} \underbrace{[\partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n}\cdot\partial_t\boldsymbol{J})]}_{\text{the part of } \partial_t J \text{ transverse to } \boldsymbol{n}} \quad . \tag{11}$$

$$= \boldsymbol{n} \times \left[\frac{\boldsymbol{n}}{c} \times \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{1}{c}\frac{\partial \boldsymbol{J}(T, r_o)}{\partial T}\right] \tag{12}$$

$$\tag{13}$$

Problem 2. Dipole Fields

Consider a small ectric dipole with harmonic time dependence, $\boldsymbol{p}(t) = \boldsymbol{p}_o e^{-i\omega t}$. Recall that in homework 6 we determined the electric field through order ω^2 in frequency using a quasi-static near field expansion

$$\boldsymbol{E}(t) = \frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}(t)) - \boldsymbol{p}(t)}{4\pi r^2} - \frac{\boldsymbol{n}(\boldsymbol{n}\cdot\ddot{\boldsymbol{p}}) + \ddot{\boldsymbol{p}}}{8\pi r c^2} \tag{14}$$

The purpose of the problem is to examine the transition to the far field, by computing the exact electric field as a function of radius.

(a) Define near and far field. Express your results in terms of the wave number $k = \omega/c$.

(b) Start from the exact expressions

$$\varphi(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\rho(T,\boldsymbol{r}_o)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{15}$$

$$\boldsymbol{A}(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\boldsymbol{J}(T,\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{16}$$

where

$$T = t - \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c} \simeq t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}\,, \tag{17}$$

and assume harmonic time dependence of $\rho(t_o,\boldsymbol{r}_o) = \rho(\boldsymbol{r}_o)e^{-i\omega t_o}$ and $\boldsymbol{J}(t_o, r_o) = \boldsymbol{J}(r_o)e^{-i\omega t}$, without making a far field expansion show that

$$\varphi(t,\boldsymbol{r}) = \frac{e^{-i\omega t+ikr}}{4\pi r^2}\boldsymbol{n}\cdot\boldsymbol{p}_o\,(1-ikr) \tag{18}$$

$$\boldsymbol{A}(t,\boldsymbol{r}) = -ik\frac{e^{-\omega t+ikr}}{4\pi r}\boldsymbol{p}_o \tag{19}$$

(c) Show by direct differentiation of the potentials $\boldsymbol{A}$ and φ that in the far field you recover the result given in class

$$\boldsymbol{E} = \frac{k^2 e^{-i\omega t+ikr}}{4\pi r}\left[-\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_o\right] \tag{20}$$

Comment on all qualitative features.

When differentiating, note carefully the contribution from $\boldsymbol{A}$ and φ, and how they conspire to make a field $\boldsymbol{E}$ which is transverse to $\boldsymbol{n}$.

(d) Show that in general

$$\boldsymbol{E}(t,r) = e^{-i\omega t+ikr}\left[\frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}_o) - \boldsymbol{p}_o}{4\pi r^3} - ik\frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}_o) - \boldsymbol{p}_o}{4\pi r^2} + k^2\frac{-\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_o}{4\pi r}\right] \tag{21}$$

Comment on all qualitative features. Write Eq. (21) in coordinate space expressed in terms of $\boldsymbol{p}(t_e)$, $\dot{\boldsymbol{p}}(t_e)$, and $\ddot{\boldsymbol{p}}(t_e)$ with $t_e = t - r/c$ (compare to the typed course notes sec. 11.3), and show consistency of this result with the near field result derived with quasi-statics.

Problem 3. Radiation in the lowest Bohr Orbit

In the Bohr model, a classical non-relativistic electron circles a proton in a circular orbit with angular momentum $L = \hbar$, due to the Coulomb attraction between the electron and the proton.

(a) Recall that that the electron kinetic energy is half of minus its potential energy (for a coulomb orbit). Recall also that the lowest bohr orbit has velocity, $\beta = \alpha$ where $\beta = v_e/c$, and $\alpha = e^2/(4\pi\hbar c) = 1/137$. Prove these statements.

(b) Write down the (total=kinetic + potential) energy and radius of the lowest Bohr orbit in terms of the electron mass, m_e, $\hbar$, c and α. What is the size of the Bohr radius a_o compared to the electron compton wavelength, *i.e.* $a_o/(\hbar/(m_e c))$?

(c) One of the difficulties with the Bohr model, is that classically the electron would radiate. Determine the energy lost to radiation per unit time, for an electron in the lowest orbit.

(d) Determine the energy radiated per revolution in the Bohr model, ΔE, and compare ΔE to the (kinetic+potential) energy of the orbit, *i.e.* compute $\Delta E/E_{\rm orbit}$. Express $\Delta E/E_{\rm orbit}$ in terms of the fine structure constant, and estimate its value.

(e) If the electron moves in the x, y plane determine the time averaged power radiated per solid angle, $\overline{dP}/d\Omega$. Use a complex notation $\boldsymbol{r}(t) = a_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\omega_o t}$.

You should find

$$\frac{\overline{dP}}{d\Omega} = \frac{e^2}{16\pi^2 c^3}\,\frac{1}{2}(1+\cos^2\theta)\,(\omega_o^2 a_o)^2 \tag{22}$$

where ω_o is the angular velocity of the electron

(f) Check your result of part (e) by integrating over solid angle and comparing with part (c).

(g) Now we will study the polarization of the light. (These questions do not require calculation).

(i) If the emitted light is observed along x axis, what is the polarization of the radiated light? Explain physically.

(ii) If the emitted light is observed along the y axis, what is the polarization of the radiated light? Explain physically.

(iii) If the emitted light is observed along the z axis, what is the polarization of the light? Explain physically.

(h) The power radiated along the z-axis is twice as large as the power radiated along the x-axis. Explain this result physically.

Problem 4. Radiation from a Phased Array

A current distribution consists of N identical souces. The k-th source is identical to the first source except for a rigid translation by an amount $\boldsymbol{R}_k$ ($k = 1, 2, \ldots, N$). The sources oscillate at the same frequency but have different phases δ_k. That is

$$\boldsymbol{j}_k \propto \exp\left(-i(\omega t + \delta_k)\right) \tag{23}$$

(a) Show that the angular distribution of radiated power can be written as a product of two factors: one is the angular distribution for $N = 1$; the other depends on $\boldsymbol{R}_k$ and δ_k, but not on the structure of the sources.

(b) The planes of two square loops (each with sided length a) are centered on (and lie perpendicular to) the z-axis at $z = \pm a/2$. The loop edges are parrallel to the x and y coordinate axes. Find the angular distribution of power in the $x-z$ plane if the current at all points in both loops is $I\cos(\omega t)$. Make a polar plot of the angular distribution of power for $\omega c/a = 2\pi$ and $\omega c/a \ll 1$. Identify the multipole character of the radiation in the limit $\omega a/c \ll 1$.

You should find

$$\frac{dP}{d\Omega} = \frac{I_o^2 a^2 \omega^2}{32\pi^2 c^3} \left(2\sin(\sin\theta ka/2)\right)^2 \left(2\cos(\cos\theta ka/2)\right)^2 \tag{24}$$

(c) The limit $\omega a/c \ll 1$ has a simple physical interpretation. Describe this interpretation and show that it reproduces all aspects of the power distribution (including normalization factors) in the limit $\omega a/c \ll 1$.

(d) Repeat part (b) when the current in the upper loop is $I\cos\omega t$ and the current in the lower loop is $-I\cos\omega t$.

Problem 5. A Charged Rotor: Zangwill

Two identical point charges of charge q are fixed to the ends of a rod of length 2ℓ which rotates with constant angular velocity of $\frac{1}{2}\omega$ in the $x - y$ plane about an axis perpendicular to the rod and through its center

(a) Calculate the electrid dipole moment $\boldsymbol{p}(t)$. Is there electric dipole radiation?

(b) Calculate the magnetic dipole moment $\boldsymbol{m}(t)$. Is there magnetic dipole radiation?

(c) Show that the electric quadruple moment is

$$Q(t) = 3q\ell^2 \begin{pmatrix} \frac{1}{3} + \cos\omega t & \sin\omega t & 0 \\ \sin\omega t & \frac{1}{3} - \cos\omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \tag{25}$$

(d) Show that the time averaged angular distribution of radiated power is

$$\overline{\frac{dP}{d\Omega}} = \frac{1}{128\pi^2 c^5}\omega^6 q^2 \ell^4 (1 - \cos^4\theta)\,. \tag{26}$$

Problem 6. Basics of Relativity

(a) (Optional) The space time event at $X^\mu = (X^0, X^i) = (ct, \boldsymbol{x})$ happens at $\underline{X}^\mu = (\underline{X}^0, \underline{X}^i) = (c\underline{t}, \underline{\boldsymbol{x}})$ according to an observer moving to the right along the x axis with velocity v. Define the "light-cone" coordinates $x^+ \equiv X^0 + X^1$ and $x^- \equiv X^0 - X^1$. Show that under this boost that the x^+ coordinates are contracted, while the x^- coordinates are elongated

$$\underline{x}^+ = e^{-y} x^+ = \sqrt{\frac{1-\beta}{1+\beta}} x^+ \,, \tag{27}$$

$$\underline{x}^- = e^{y} x^- = \sqrt{\frac{1+\beta}{1-\beta}} x^- \,. \tag{28}$$

Here

$$y = \tanh^{-1}\beta = \frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right) \tag{29}$$

is the so-called "rapidity" of the boost. What is $\underline{x}^+\underline{x}^-$ and why is it unchanged under boost?

(b) (Optional) A Lorentz tensor transforms as

$$\underline{T}^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma T^{\rho\sigma} \tag{30}$$

Show that the transformation rule can be alternatively written

$$\underline{T}^\mu{}_\nu = (\mathcal{L})^\mu{}_\rho T^\rho{}_\sigma (\mathcal{L}^{-1})^\sigma{}_\nu \tag{31}$$

or equivalently

$$\underline{T}^\mu{}_\nu = L^\mu{}_\rho L_\nu{}^\sigma T^\rho{}_\sigma \tag{32}$$

(c) (Optional) The frequency and wave number of a plane wave of light, $e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{iK\cdot X}$, form a lightlike four vector

$$K^\mu = \left(\frac{\omega}{c}, \boldsymbol{k}\right) \tag{33}$$

(i) Show that $K \cdot K = K_\mu K^\mu = 0$ (this is the statement that K is lightlike.)

(ii) If a photon has frequency ω_o and is propagating along the z-axis, show (using the 4-vector properties of K^μ) that according to an observer propagating in the negative z direction with speeed β

$$\omega = \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_o \tag{34}$$

(d) (Optional) Show that the four velocity $U^\mu = dx^\mu/d\tau$ satisfies $U_\mu U^\mu = -c^2$.

(e) (Optional) For a particle with four momentum $P^\mu = (\frac{E}{c}, \boldsymbol{p}) = mU^\mu$ show that $P_\mu P^\mu = -(mc^2)^2/c^2$. This determines $E(\boldsymbol{p})$ the relation between energy and momentum:

$$\frac{E(\boldsymbol{p})}{c} = \sqrt{\boldsymbol{p}^2 + (mc)^2}\,. \tag{35}$$

(i) Show the velocity of the particle (i.e. the group velocity) is

$$\boldsymbol{v_p} \equiv \frac{\partial E(\boldsymbol{p})}{\partial \boldsymbol{p}} = \frac{c^2 \boldsymbol{p}}{E} \tag{36}$$

(f) (Do me! Not optional) A particle with velocity v_p in the x direction. Using the 4-vector transformation properties of U^μ, show that according to an observer moving to the right with velocity v, the particle moves with velocity

$$\underline{v}_p = \frac{v_p - v}{1 - v_p v/c^2} \tag{37}$$

Problem 7. One liners

(a) Starting from the Maxwell equations for $F^{\mu\nu}$ and the definition of $F^{\mu\nu}$, derive the wave equation $-\Box A^\mu = J^\mu/c$.

(b) Starting from the maxwell equations for $F^{\mu\nu}$ in covariant form, show that we must have $\partial_\mu J^\mu = 0$ for consistency.

(c) (This is two lines) Show that the energy conservation and force laws

$$\frac{dE_{\boldsymbol{p}}}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}_p \tag{38}$$

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \frac{\boldsymbol{v_p}}{c}\times\boldsymbol{B}) \tag{39}$$

can be written covariantly

$$\frac{dP^\mu}{d\tau} = F^{\mu\nu}u_\nu/c \tag{40}$$

Note that $E_{\boldsymbol{p}}$ (the energy of the particle) is different from $\boldsymbol{E}$ the electric field.

(d) From Eq. (40) show that $P_\mu P^\mu$ is constant in time.

(e) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transform

$$A_\mu \to A_\mu + \partial_\mu \Lambda(X) \tag{41}$$

where Λ is an arbitrary function of $X = (t, \boldsymbol{r})$.

(f) Given $F^{\mu\nu}$ the only two Lorentz invariant quantities are $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Evaluate these two invariants in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$[1]

[1] answers: $2(B^2 - E^2)$ and $-4E\cdot B$