See "Technical note on computing radiation fields"

Problem 1. (Optional) Electric field in the far field

If you get stuck check the notes online. The scalar and vector potential in the far field are

$$\varphi(t, \boldsymbol{r}) = \frac{1}{4\pi r} \int d^3\boldsymbol{r}_o \rho(T, \boldsymbol{r}_o) \tag{1}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{1}{4\pi r} \int d^3\boldsymbol{r}_o \boldsymbol{J}(T, \boldsymbol{r}_o)/c \tag{2}$$

where the retarded time $T = t - |\boldsymbol{r} - \boldsymbol{r}_o|/c$ in the far field is

$$T = t - r/c + \frac{\boldsymbol{n} \cdot \boldsymbol{r}_o}{c} \tag{3}$$

The goal is to compute the electric field

$$\boldsymbol{E}(t, r) = -\frac{1}{c}\partial_t \boldsymbol{A}(t, \boldsymbol{r}) - \nabla\varphi(t, \boldsymbol{r}) \tag{4}$$

(a) (Optional) Consider the change of variable $t, \boldsymbol{r}_o \to T, \boldsymbol{r}_o$. Show that

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} \tag{5}$$

$$\left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_T = \left(\frac{\partial}{\partial \boldsymbol{r}_o}\right)_t - \frac{\boldsymbol{n}}{c}\frac{\partial}{\partial t} \tag{6}$$

(b) (Optional) Compute

$$\left(\frac{\partial}{\partial t} + c\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) T \quad =0 \tag{7}$$

You should find a simple result. Interpret the answer using the definition of T

$T \equiv$ the time when the light should be emitted from $\boldsymbol{r}_o$ to arrive at the observation point $(t, \boldsymbol{r})$.

How do you interpert the derivative:

$$\left(\frac{\partial}{\partial t} + c\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) \tag{8}$$

It is the derivative moving with the light. If you change the observation point (t,r) to (t + dt, r + c n dr) the light (the emitted at (T,r0) will reach both of these points without changing T or r0.

(c) (Optional) Show that

$$\boldsymbol{E} = -\frac{1}{4\pi r c^2} \int_{r_o} \frac{\partial \boldsymbol{J}(T, r_o)}{\partial T} + \frac{\boldsymbol{n}}{c}\frac{1}{4\pi r} \int_{r_o} \frac{\partial \rho(T, r_o)}{\partial T} \tag{9}$$

(d) (Optional) Use current conservation to express

$$\frac{\partial \rho(T, \boldsymbol{r}_o)}{\partial T} = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_T = -\left(\nabla_{r_o} \cdot \boldsymbol{J}\right)_t + \frac{\boldsymbol{n}}{c} \cdot \frac{\partial \boldsymbol{J}}{\partial T} \tag{10}$$

where $(\nabla_{r_o} \cdot \boldsymbol{J})_t$ denotes the divergence at fixed observation time

(e) (Optional) Conclude that only the transverse piece of the current to $\boldsymbol{n}$ contributes to the radiation field

$$\boldsymbol{E} = -\frac{1}{4\pi r}\frac{1}{c^2}\int_{\boldsymbol{r}_o} \underbrace{\left[\partial_t \boldsymbol{J} - \boldsymbol{n}(\boldsymbol{n}\cdot\partial_t\boldsymbol{J})\right]}_{\text{the part of } \partial_t J \text{ transverse to } \boldsymbol{n}} \quad . \tag{11}$$

$$= \boldsymbol{n} \times \left[\frac{\boldsymbol{n}}{c} \times \frac{1}{4\pi r}\int_{\boldsymbol{r}_o} \frac{1}{c}\frac{\partial \boldsymbol{J}(T, r_o)}{\partial T}\right] \tag{12}$$

$$\tag{13}$$

Problem 2. Dipole Fields

Consider a small ectric dipole with harmonic time dependence, $\boldsymbol{p}(t) = \boldsymbol{p}_o e^{-i\omega t}$. Recall that in homework 6 we determined the electric field through order ω^2 in frequency using a quasi-static near field expansion

$$\boldsymbol{E}(t) = \frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}(t)) - \boldsymbol{p}(t)}{4\pi r^2} - \frac{\boldsymbol{n}(\boldsymbol{n}\cdot\ddot{\boldsymbol{p}}) + \ddot{\boldsymbol{p}}}{8\pi r c^2} \tag{14}$$

The purpose of the problem is to examine the transition to the far field, by computing the exact electric field as a function of radius.

(a) Define near and far field. Express your results in terms of the wave number $k = \omega/c$.

(b) Start from the exact expressions

$$\varphi(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\rho(T,\boldsymbol{r}_o)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{15}$$

$$\boldsymbol{A}(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \frac{\boldsymbol{J}(T,\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|} \tag{16}$$

where

$$T = t - \frac{|\boldsymbol{r}-\boldsymbol{r}_o|}{c} \simeq t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_o}{c}\,, \tag{17}$$

and assume harmonic time dependence of $\rho(t_o,\boldsymbol{r}_o) = \rho(\boldsymbol{r}_o)e^{-i\omega t_o}$ and $\boldsymbol{J}(t_o,r_o) = \boldsymbol{J}(r_o)e^{-i\omega t}$, without making a far field expansion show that

$$\varphi(t,\boldsymbol{r}) = \frac{e^{-i\omega t + ikr}}{4\pi r^2}\boldsymbol{n}\cdot\boldsymbol{p}_o\,(1-ikr) \tag{18}$$

$$\boldsymbol{A}(t,\boldsymbol{r}) = -ik\frac{e^{-\omega t + ikr}}{4\pi r}\boldsymbol{p}_o \tag{19}$$

(c) Show by direct differentiation of the potentials $\boldsymbol{A}$ and φ that in the far field you recover the result given in class

$$\boldsymbol{E} = \frac{k^2 e^{-i\omega t + ikr}}{4\pi r}\left[-\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_o\right] \tag{20}$$

Comment on all qualitative features.

When differentiating, note carefully the contribution from $\boldsymbol{A}$ and φ, and how they conspire to make a field $\boldsymbol{E}$ which is transverse to $\boldsymbol{n}$.

(d) Show that in general

$$\boldsymbol{E}(t,r) = e^{-i\omega t + ikr}\left[\frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}_o) - \boldsymbol{p}_o}{4\pi r^3} - ik\frac{3\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{p}_o) - \boldsymbol{p}_o}{4\pi r^2} + k^2\frac{-\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{p}_o}{4\pi r}\right] \tag{21}$$

Comment on all qualitative features. Write Eq. (21) in coordinate space expressed in terms of $\boldsymbol{p}(t_e)$, $\dot{\boldsymbol{p}}(t_e)$, and $\ddot{\boldsymbol{p}}(t_e)$ with $t_e = t - r/c$ (compare to the typed course notes sec. 11.3), and show consistency of this result with the near field result derived with quasi-statics.

Dipole Fields

$\boxed{a)}$ We mean $r \ll w/c$ for near field and $r \gg w/c$ for far field.

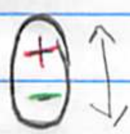
$\boxed{b)}$ Then

$$\varphi(t, r) = \int d^3 r_0 \, \frac{\rho(T, r_0)}{4\pi |\vec{r} - \vec{r}_0|} \quad \text{where}$$

$$T = t - \frac{|\vec{r} - \vec{r}_0|}{c}$$

- Imagine a dipole

• $\vec{r}$



$\vec{r}_0$ integrates over this region, $\underline{\vec{r} \gg \vec{r}_0}$

Then

- $|\vec{r} - \vec{r}_0| \simeq r - \vec{n} \cdot \vec{r}_0$
- $\dfrac{1}{|r - \vec{r}_0|} \simeq \dfrac{1}{r}\left(1 + \dfrac{\vec{n} \cdot \vec{r}_0}{r}\right)$
- $T = t - \dfrac{|\vec{r} - \vec{r}_0|}{c} \simeq t - \dfrac{r}{c} + \dfrac{\vec{n} \cdot \vec{r}_0}{c}$

For this problem $\rho(T, \vec{r}_0) = \rho(r_0)\, e^{-i\omega T}$

So

expand me!

$$\varphi(t,r) = \int \frac{d^3 r_0}{4\pi r} \vec{\rho}(\vec{r}_0)\, e^{-i\omega(t - r/c + \frac{\vec{n}\cdot\vec{r}_0}{c})} \left(1 + \frac{\vec{n}\cdot\vec{r}_0}{r}\right)$$

$$\varphi(t,r) \simeq \frac{1}{r}\int \frac{d^3 r_0}{4\pi} \rho(r_0)$$

This is zero! There is no net charge.

$$\frac{e^{-i\omega(t - r/c)}}{4\pi r}\int d^3 r_0\, \rho(r_0)\left(\frac{\vec{n}\cdot\vec{r}_0}{r} - ik\,\vec{n}\cdot r_0\right)$$

- Since

$$\int d^3 r_0\, \rho(r_0)\, \vec{r}_0 = \vec{p}_0$$

We find

$$\varphi(t,r) = \frac{e^{-i\omega(t - r/c)}}{4\pi r}\left[\frac{\vec{n}\cdot\vec{p}_0}{r} - ik\,\vec{n}\cdot\vec{p}_0\right]$$

Similarly

$$\vec{A}(t,r) = \int d^3\vec{r}_0\, \frac{\vec{J}(T, \vec{r}_0)/c}{4\pi|\vec{r} - \vec{r}_0|}$$

- Let us work to lowest order. Previously we couldn't do this because the zeroth order term vanished on the previous page

$$\vec{A}(t, r) \simeq \frac{1}{4\pi r} \int d^3 r_0 \, \frac{\vec{J}(T, r_0)}{c}$$

Recall that

$$\int d^3 r_0 \, \vec{J}(T, r_0) = \frac{\partial}{\partial T} \vec{p}(T, r_0)$$

i.e. in a dipole approx, with the dipole at the origin

$$\vec{J}(t, r_0) = \partial_t \vec{p}(t) \, \delta^3(\vec{r}_0)$$

So

$$\vec{A} = \frac{1}{4\pi r c} \frac{\partial}{\partial T} \vec{p}(T)$$

Now

$$\vec{p}(T) = \vec{p}_0 \, e^{-i\omega T}$$

$$\frac{\partial \vec{p}}{\partial T} = -i\omega \, \vec{p}_0 \, e^{-i\omega T}$$

$$\vec{A} \simeq \frac{-i\omega \, p_0 \, e^{-i\omega T}}{4\pi r}$$

So to zeroth order

$$T = t - \frac{r}{c} + \cancel{\frac{n \cdot \vec{r}_0}{c}} \xrightarrow{} 0 \quad \text{at zeroth order}$$

and

$$\vec{A} = \frac{-i\omega\,\vec{p}_0}{4\pi r c}\, e^{-i\omega(t - r/c)}$$

The answer follows using $k = \omega/c$

$\boxed{c)}$ In the far field $rk \gg 1$

- $$\varphi = \frac{e^{-i\omega t + ikr}}{4\pi r}\left(-ik\,\vec{n}\cdot\vec{p}_0\right)$$

- $$\vec{A} = \frac{e^{-i\omega t + ikr}}{4\pi r}\left(-ik\,\vec{p}_0\right)$$

- Then

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

where in the far field the spatial derivatives act on the phase. If they acted on $\frac{1}{r}$

you would get something which is not far field:

$$\nabla\left(\frac{1}{r}\right) \propto \frac{1}{r^2} \quad \text{falls off too fast as } r \to \infty$$

- So

$$\vec{E} = \frac{e^{-i\omega t+ikr}}{4\pi r}\left(+\frac{i\omega}{c}\left(-ik\,\vec{p}_0\right)\right)$$

$$+ \frac{e^{-i\omega k+ikr}}{4\pi r}\left(\left(-ik\,\vec{n}\right)\left(-ik\,\vec{n}\cdot\vec{p}_0\right)\right)$$

Thus

$$= -k^2\,\frac{e^{-i\omega t+ikr}}{4\pi r}\left(\underbrace{-\vec{p}_0}_{\text{from } -\frac{1}{c}\partial_t \vec{A}} + \underbrace{\vec{n}\,(n\cdot p_0)}_{\text{from } -\nabla\varphi}\right)$$

$$\vec{E} = -k^2\,\frac{e^{-i\omega t+ikr}}{4\pi r}\left(\vec{n}\times\vec{n}\times\vec{p}_0\right)$$

Watch me!!

- We notice that it is the $-\nabla\varphi$ term which cancels the piece of $-\frac{1}{c}\partial_t\vec{A}$ which is parallel to $\vec{n}$. The $\vec{E}$-field must be transverse to $\vec{n}$ in the far field. Current conservation relates φ and A and guarantees that this will always happen.

d) Algebra:

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\varphi$$ General resut

$$\varphi = \frac{e^{-i\omega t + ikr}}{4\pi r}(\vec{n}\cdot\vec{p}_0)\left(\frac{1}{r} - ik\right)$$

$$\vec{A} = \frac{e^{-i\omega t + ikr}}{4\pi r}(-ik\,\vec{p}_0)$$

From previous item

- $$-\frac{1}{c}\partial_t \vec{A} = \frac{e^{-i\omega t + ikr}}{4\pi r}(+k^2\,\vec{p}_0)$$

derivative of phase

- $$-\nabla\varphi = \frac{e^{-i\omega t + ikr}}{4\pi}(n\cdot p_0)\left(\frac{1}{r^2} - \frac{ik}{r}\right)\left(-ik\,\vec{n}\right)$$

$$+ \frac{e^{-i\omega t + ikr}}{4\pi}(\vec{n}\cdot p_0)\left(\frac{2}{r^3}\vec{n} \mp \frac{ik}{r^2}\vec{n}\right)$$

derivative of $1/r$

$$+ \frac{e^{-i\omega t + ikr}}{4\pi}\left(\frac{1}{r^2} - \frac{ik}{r}\right)\left(\frac{-\vec{p} + \vec{n}(n\cdot\vec{p})}{r}\right)$$

derivative of $(\vec{n}\cdot\vec{p})$

$$\frac{\partial}{\partial r^i}\left(\frac{r_j}{r}\right) = \delta_{ij} - n_i n_j$$

Collecting terms in powers of $1/r$

$$\vec{E} = \frac{e^{-i\omega t + ikr}}{4\pi}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{p}_o) - \vec{p}_o}{r^3} - ik\,\frac{3\vec{n}(\vec{n}\cdot\vec{p}_o) - \vec{p}_o}{r^2} + k^2\,\frac{(\vec{p}_o - \vec{n}(n\cdot p_o))}{r}\right]$$

So in time

$$\boxed{\vec{E} = \frac{1}{4\pi}\left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\left(\frac{3\vec{n}(\vec{n}\cdot\vec{p}(t_e)) - \vec{p}(t_e)}{r^3}\right) + \frac{\vec{n}\times\vec{n}\times\ddot{\vec{p}}(t_e)}{4\pi r c^2}\right] \qquad t_e \equiv t - r/c}$$

- Note the $\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)$ structure is guaranteed by the fact that in quasi-statics the electric field corrects start like $\frac{1}{c^2}$

- Expanding $t_e = t - r/c$ in powers of r/c

$$\vec{E} = \frac{3\vec{n}(\vec{n}\cdot\vec{p}(t)) - \vec{p}(t)}{4\pi r^3} + \underset{\substack{\uparrow \\ \text{cancellation} \\ \text{at order } 1/c\,!}}{0} + O\left(\frac{1}{c^2}\right)$$

The $1/c^2$ term is:

$$-\frac{1}{2}\left(\frac{r}{c}\right)^2\left(\frac{3\vec{n}\,(\vec{n}\cdot\ddot{\vec{p}}(t))-\ddot{\vec{p}}}{4\pi r^3}\right)+\frac{-\ddot{\vec{p}}+\vec{n}\,(\vec{n}\cdot\ddot{\vec{p}})}{4\pi r c^2}$$

$$=-\left(\frac{\ddot{\vec{p}}+\vec{n}\,(\vec{n}\cdot\ddot{\vec{p}})}{4\pi r c^2}\right)$$

in agreement with our quasistatic result.

Problem 3. Radiation in the lowest Bohr Orbit

In the Bohr model, a classical non-relativistic electron circles a proton in a circular orbit with angular momentum $L = \hbar$, due to the Coulomb attraction between the electron and the proton.

(a) Recall that that the electron kinetic energy is half of minus its potential energy (for a coulomb orbit). Recall also that the lowest bohr orbit has velocity, $\beta = \alpha$ where $\beta = v_e/c$, and $\alpha = e^2/(4\pi\hbar c) = 1/137$. Prove these statements.

(b) Write down the (total=kinetic + potential) energy and radius of the lowest Bohr orbit in terms of the electron mass, m_e, $\hbar$, c and α. What is the size of the Bohr radius a_o compared to the electron compton wavelength, *i.e.* $a_o/(\hbar/(m_e c))$?

(c) One of the difficulties with the Bohr model, is that classically the electron would radiate. Determine the energy lost to radiation per unit time, for an electron in the lowest orbit.

(d) Determine the energy radiated per revolution in the Bohr model, ΔE, and compare ΔE to the (kinetic+potential) energy of the orbit, *i.e.* compute $\Delta E/E_{\rm orbit}$. Express $\Delta E/E_{\rm orbit}$ in terms of the fine structure constant, and estimate its value.

(e) If the electron moves in the x, y plane determine the time averaged power radiated per solid angle, $\overline{dP}/d\Omega$. Use a complex notation $\boldsymbol{r}(t) = a_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\omega_o t}$.

You should find
$$\frac{\overline{dP}}{d\Omega} = \frac{e^2}{16\pi^2 c^3}\,\frac{1}{2}(1+\cos^2\theta)\,(\omega_o^2 a_o)^2 \tag{22}$$
where ω_o is the angular velocity of the electron

(f) Check your result of part (*e*) by integrating over solid angle and comparing with part (*c*).

(g) Now we will study the polarization of the light. (These questions do not require calculation).

  (i) If the emitted light is observed along x axis, what is the polarization of the radiated light? Explain physically.

  (ii) If the emitted light is observed along the y axis, what is the polarization of the radiated light? Explain physically.

  (iii) If the emitted light is observed along the z axis, what is the polarization of the light? Explain physically.

(h) The power radiated along the z-axis is twice as large as the power radiated along the x-axis. Explain this result physically.

a) Optional

For the Bohr orbit,

$$F = m\frac{v^2}{r}$$

$$\frac{e^2}{4\pi r^2} = \frac{mv^2}{r}$$

$$\frac{e^2}{4\pi r} = mv^2 \xrightarrow{\text{So}} (-PE) = 2\,KE$$

So,

$$\frac{+e^2}{4\pi a_0} = 2\cdot\left(\frac{\hbar^2}{2ma_0^2}\right) \longleftarrow \text{the KE is } \frac{\hbar}{2ma_0^2}$$

$$a_0 = \frac{\hbar^2}{m\left(\frac{e^2}{4\pi}\right)} = \frac{\hbar}{mc\left(\frac{e^2}{4\pi\hbar c}\right)}$$

$$a_0 = \frac{\hbar}{mc\,\alpha},$$

$$L = mvr$$

$$\frac{\hbar}{ma_0} = V \implies V = \alpha c$$

Then

b) So

$$E = KE + PE = -\frac{1}{2} PE + PE = \frac{PE}{2}$$

Then,

$$E = \frac{-e^2}{2(4\pi a_0)} = \frac{1}{2} mc^2 \alpha^2$$

$$a_0 = \frac{\hbar}{mc\alpha}$$

$$\left(\frac{a_0}{\hbar/mc}\right) = \frac{1}{\alpha} = 137$$

Bohr

a) $$a_0 = \frac{\hbar}{mc\alpha}$$

$$E = -\frac{1}{2}\frac{e^2}{4\pi a_0}$$

b) Using the Larmour formula

$$a = \omega_0^2 a_0$$

$$P = \frac{e^2}{4\pi}\frac{2}{3}\frac{a^2}{c^3}$$

☆ $$\boxed{P = \frac{e^2}{4\pi}\frac{2}{3}\frac{\omega_0^4 a_0^2}{c^3}}$$ $$\omega_0 = \frac{\alpha c}{a_0}$$

$$P = \frac{e^2}{4\pi}\frac{2}{3}\frac{\alpha^4 c}{a_0^2}$$

☆☆ $$\boxed{P = \frac{e^2}{4\pi a_0}\frac{2}{3}\alpha^4\frac{c}{a_0}}$$

c) So for part .

$$\Delta E = \frac{2\pi a_0}{\alpha c} P$$

So
$$\Delta E = \frac{e^2}{4\pi a_0} \frac{2}{3} \alpha^4 \frac{c}{a_0} \frac{2\pi a_0}{c\alpha}$$

$$\Delta E = \frac{e^2}{4\pi a_0} \frac{4\pi}{3} \cdot \alpha^3$$

And
$$\boxed{\frac{\Delta E}{E} = \frac{8\pi}{3} \alpha^3} \sim 10^{-6}$$

d) Using the Larmour result

$$E_{rad} = \frac{e}{4\pi r c^2} n \times n \times a(t_e)$$

Using
$$\vec{r} = (\cos \omega_0 t_e, \sin \omega_0 t_e, 0) a_0$$

$$a = -\omega_0^2 a_0 (\cos \omega_0 t, \sin \omega_0 t, 0)$$

We $\Longrightarrow$
$$\vec{a} = -\omega_0^2 a_0 (\hat{x} e^{-i\omega_0 t} + i\hat{y} e^{-i\omega_0 t})$$

will take the
$$\vec{a} = -\omega_0^2 a_0 e^{-i\omega t} (\hat{x} + i\hat{y})$$

real part of this

So then

$$\frac{d\bar{P}}{d\Omega} = \frac{e^2}{16\pi c^3} \frac{|\vec{n}\times\vec{n}\times\vec{a}|^2}{2}$$

$$n\times n\times\vec{a} = -\vec{a} + \vec{n}(n\cdot\vec{a})$$

So taking $\vec{n}$ in the x, z plane:

$$n\times n\times\vec{a} \propto -(\hat{x}+i\hat{y}) + \vec{n}(\vec{n}\cdot(\hat{x}+i\hat{y}))$$

$$\propto -(\hat{x}+i\hat{y}) + \vec{n}(n\cdot\hat{x})$$

$$|n\times n\times\vec{a}|^2 \propto (\hat{x}^2+\hat{y}^2) - (n\cdot x)^2$$

$$\propto 2 - \sin^2\theta$$

Then

$$\frac{d\bar{P}}{d\Omega} = \frac{e^2}{16\pi c^3}\frac{1}{2}(2-\sin^2\theta)(\omega_0^2 a_0)^2$$

Problem 4. Radiation from a Phased Array

A current distribution consists of N identical souces. The k-th source is identical to the first source except for a rigid translation by an amount $\boldsymbol{R}_k$ ($k = 1, 2, \ldots, N$). The sources oscillate at the same frequency but have different phases δ_k. That is

$$\boldsymbol{j}_k \propto \exp\left(-i(\omega t + \delta_k)\right) \tag{23}$$

(a) Show that the angular distribution of radiated power can be written as a product of two factors: one is the angular distribution for $N = 1$; the other depends on $\boldsymbol{R}_k$ and δ_k, but not on the structure of the sources.

(b) The planes of two square loops (each with sided length a) are centered on (and lie perpendicular to) the z-axis at $z = \pm a/2$. The loop edges are parrallel to the x and y coordinate axes. Find the angular distribution of power in the $x-z$ plane if the current at all points in both loops is $I\cos(\omega t)$. Make a polar plot of the angular distribution of power for $\omega c/a = 2\pi$ and $\omega c/a \ll 1$. Identify the multipole character of the radiation in the limit $\omega a/c \ll 1$.

You should find

$$\frac{dP}{d\Omega} = \frac{I_o^2 a^2 \omega^2}{32\pi^2 c^3} \left(2\sin(\sin\theta ka/2)\right)^2 \left(2\cos(\cos\theta ka/2)\right)^2 \tag{24}$$

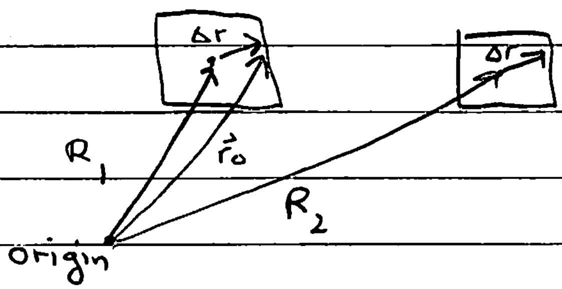
(c) The limit $\omega a/c \ll 1$ has a simple physical interpretation. Describe this interpretation and show that it reproduces all aspects of the power distribution (including normalization factors) in the limit $\omega a/c \ll 1$.

(d) Repeat part (b) when the current in the upper loop is $I\cos\omega t$ and the current in the lower loop is $-I\cos\omega t$.

Radiation from a phased Array

a) $$r\vec{A}_{rad} = \frac{1}{4\pi}\int_{r_0} \frac{\vec{J}}{c}(T, \vec{r}_0)$$

The integration over the current is as follows:



$\vec{R}_a$ points from origin to center of square

- For each element we write
$$\vec{r}_0 = \vec{R}_a + \Delta\vec{r}$$

- $$\vec{J}_a(T, \Delta r) = e^{-i\omega\left(t - \frac{r}{c} + \frac{n \cdot r_0}{c}\right)} e^{-i\delta_a} \vec{\mathcal{J}}(\Delta\vec{r})$$

current in Jth loop

canonical single square current

$$= \left(e^{-i\omega(t - \frac{r}{c})}\right)\left(e^{-i\frac{\omega}{c}\vec{n}\cdot R_a} e^{-i\delta_a}\right)\left(e^{-i\omega\vec{n}\cdot\Delta r/c}\right)\mathcal{J}(\Delta r)$$

So

$$r\vec{A}_{rad} = \frac{e^{-i\omega(t - r/c)}}{4\pi}\sum_a \int_{\Delta\vec{r}} e^{-i\omega n\cdot\Delta\vec{r}/c}\mathcal{J}(\Delta\vec{r})\, e^{-i\frac{\omega}{c}n\cdot R_a} e^{-i\delta_a}$$

$$= \left[\frac{e^{-i\omega(t - r/c)}}{4\pi}\int_{\Delta\vec{r}} e^{-i\omega\vec{n}\cdot\Delta\vec{r}/c}\vec{\mathcal{J}}(\Delta r)\right]\left[\sum_a e^{-i\frac{\omega}{c}n\cdot R_a} e^{-i\delta_a}\right]$$

$=$

Phased Array Pg. 2

So taking the Poynting flux we have

$$\frac{dP}{d\Omega} = r^2 c \Big| \underbrace{n \times n \times \frac{1}{c}\frac{\partial A_{rad}}{\partial t}}_{\vec{E}_{rad}} \Big|^2$$

We have since $e^{-i\omega(t - r/c)}$ is an irrelevant overall phase

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{c^3} \left| \frac{-i\omega}{4\pi} \int_{\Delta r_0} \vec{n} \times \vec{n} \times g(\Delta r)\, e^{-i\omega \vec{n}\cdot\Delta r/c} \right|^2 \cdot \left| \sum_a e^{-i\delta_a} e^{-i\omega \vec{n}\cdot\vec{R}_a/c} \right|^2}$$

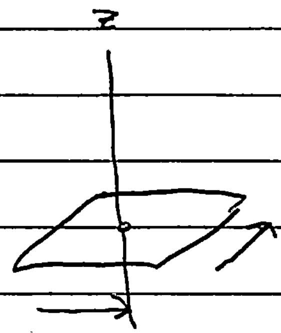
Phased Array 3

b) To work out the power from the array we must work out the integral

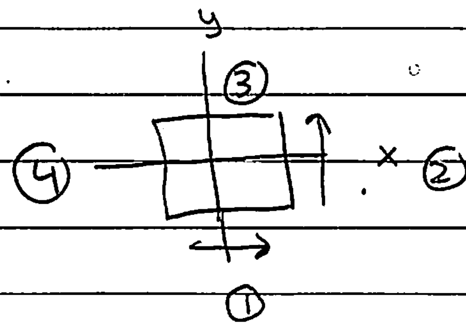
$$r\vec{A}_1 \equiv \frac{1}{4\pi}\int d^3\Delta r \, e^{-i\omega n \cdot \Delta r/c} \, \vec{J}(\Delta\vec{r})$$

from a single square loop. Taking $\vec{n}$ in the x,z plane $\vec{n} = (\sin\theta, 0, \cos\theta)$ as required by the problem statement

3D view



Top view



Lets work out the phase for each of the four legs

① $e^{-i\omega\vec{n}\cdot\Delta\vec{r}/c}$ is with $\Delta\vec{r} = (x, -\frac{a}{2}, 0)$

$$e^{-i\omega\vec{n}\cdot\Delta\vec{r}/c} = e^{-i\omega\sin\theta\, x}$$

③ Then the phase in leg 3 is also

$$e^{-i\omega\vec{n}\cdot\Delta\vec{r}/c} = e^{-i\omega\sin\theta\, x}$$

Phased Array Pg. 4

Since the phase is the same but the current is opposite

$$\int_{①+③} d^3\Delta r \; e^{-i\omega \vec{n}\cdot\vec{\Delta r}/c} \; \vec{J}(\Delta r) = 0$$

For legs ② and ④ we have with $\vec{\Delta r}_{②} = (\frac{a}{2}, y, 0)$

② $$e^{-i\omega \vec{n}\cdot\vec{\Delta r}/c} = e^{-i\frac{\omega}{c}\sin\theta \, a/2}$$

And for ④ $\vec{\Delta r}_{④} = (-a/2, y, 0)$

④ $$e^{-i\omega\vec{n}\cdot\Delta r/c} = e^{+i\frac{\omega}{c}\sin\theta \, a/2}$$

So

$$\int_{②+④} d^3\Delta r \; e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{\Delta r}} \; \vec{J} = \int dy \left(e^{-i\omega\sin\theta \, a/2} - e^{+i\omega\sin\theta \, a/2} \right) I_0 \hat{y}$$

$$= I_0 \, a \, 2i \sin\left(\frac{\omega}{c}\sin\theta \frac{a}{2}\right) \hat{y}$$

And thus

$$r\vec{A}_1 = \frac{i}{2\pi} I_0 a \sin\left(\omega \sin\theta \frac{a}{2}\right) \hat{y}$$

we need

$$\vec{n}\times\vec{n}\times(r\vec{A}_1) = -r\vec{A}_1 + \vec{n}\underbrace{(n\cdot r\vec{A}_1)}_{=0}$$

$$= -r\vec{A}_1$$

Phased Array Pg. 5

To compute the power we have

$$\frac{dP}{d\Omega} = \frac{1}{c^3}\,\omega^2\left(\frac{I_0 a}{2\pi}\sin\left(\frac{\omega\sin\theta}{c}\frac{a}{2}\right)\right)^2 \underbrace{\left|\sum_a e^{-i\delta_a} e^{-i\frac{\omega}{c}\vec{n}\cdot\vec{R}_a}\right|^2}_{\text{structure factor}}$$

So we have to work out the structure factor:

$$S.F. = \left| e^{-i\frac{\omega}{c}\cos\theta\frac{a}{2}} + e^{+i\frac{\omega}{c}\cos\theta\frac{a}{2}}\right|^2 , \qquad \vec{R}_1 = (0,0,\tfrac{a}{2}) \leftarrow \text{position of centers}$$

$$= \left|2\cos\left(\cos\theta\,\frac{ka}{2}\right)\right|^2 \qquad \vec{R}_2 = (0,0,-a/2)$$

$\delta_1 = \delta_2 = 0 \leftarrow$ phases

So the full result is

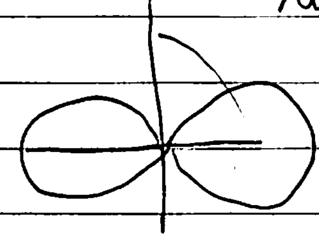
$$\boxed{\frac{dP}{d\Omega} = \frac{I_0^2 a^2 \omega^2}{16\pi^2 c^3}\left(2\sin(\sin\theta\, ka/2)\right)^2\left(2\cos(\cos\theta\, ka/2)\right)^2}$$

For $ka \ll 1$ the second factor is 4, but the first factor is expanded

$$\frac{dP}{d\Omega} = \frac{I_0^2 a^2}{16\pi^2 c^3}\,\omega^2 a^2\; 4\sin^2\theta\left(\frac{\omega a}{c}\right)^2 \cdot 4$$

$$\boxed{\frac{dP}{d\Omega} = (2\, I_0/c\; a^2)^2\,\frac{1}{16\pi^2 c^3}\,\sin^2\theta\;\omega^4}$$

$dP/d\Omega$



↳ this is the formula for magnetic dipole radiation with $\boxed{\vec{m} = 2\frac{I}{c}a^2}$ with the 2, because there are two antennas

Phased Array Pg. 6

c) If the sign of the lower current is flipped then the structure factor changes

$$S.F. = \left| e^{-i\frac{\omega}{c}\cos\theta\, a/2} - e^{i\frac{\omega}{c}\cos\theta\, a/2} \right|^2$$

$$= \left(2\sin\left(\cos\theta\ \frac{\omega a}{2c}\right)\right)^2$$

$\delta_1 = 0$, $\delta_2 = \pi$

$\vec{R}_1$, $\vec{R}_2$ Same

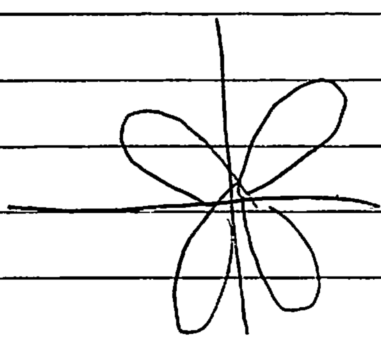
And

$$\boxed{\frac{dP}{d\Omega} = \frac{I_0^2 a^2 \omega^2}{16\pi^2 c^3} \left(2\sin\left(\sin\theta\ ka/2\right)\right)^2 \left(2\sin\left(\cos\theta\ ka/2\right)\right)^2}$$

For small frequency we find magnetic quadrupole radiation:

$$\boxed{\frac{dP}{d\Omega} = \frac{c}{16\pi^2}\left(\frac{I_0}{c}\right)^2 \cdot \left(\frac{\omega a}{c}\right)^6 \sin^2\theta\, \cos^2\theta}$$

Or



e) To check this result we integrate over Ω

$$\bar{P} = \frac{e^2}{16\pi c^3}\left(\frac{\omega_0^2 a_0}{2}\right)^2 \underbrace{\int d\Omega \, (2 - \sin^2\theta)}_{I}$$

Evaluating I:

$$I = 2\pi \int_{-1}^{1} dx \, (2 - (1 - x^2)) \qquad x \equiv \cos\theta$$

$$= 2\pi \int_{-1}^{1} dx \, (1 + x^2)$$

$$I = 2\pi \left(2 + \frac{2}{3}\right) = \frac{16\pi}{3}$$

So $$\bar{P} = \frac{e^2}{16\pi c^3}\left(\frac{\omega_0^2 a_0}{2}\right)^2 \cdot \frac{16\pi}{3}$$

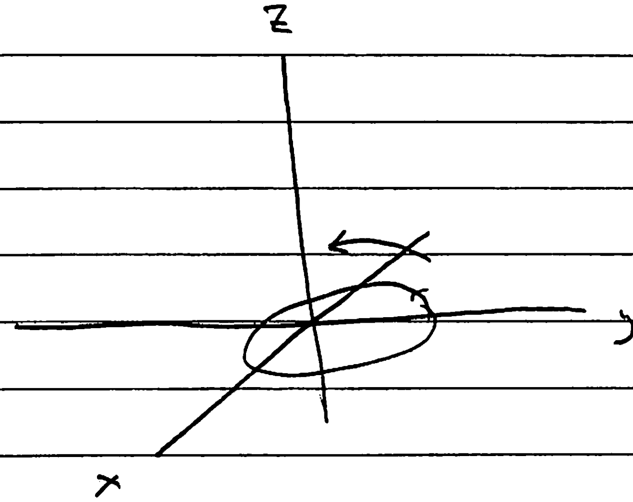
$$\bar{P} = \frac{e^2}{6\pi c^3}\left(\omega_0^2 a_0\right)^2$$ which agrees with ☆ from part b

Then the polarization is recorded by. $\vec{a}_\perp$

$$\vec{E}_{rad} = \frac{e}{4\pi r c^2} \vec{n} \times \vec{n} \times a(t_e) = \frac{e}{4\pi r c^2} \left[-\vec{a}_\perp(t_e) \right]$$

Where,

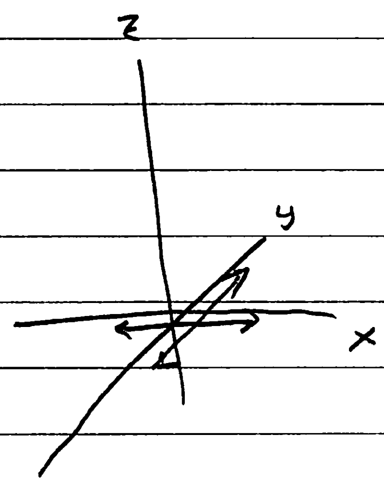
$$\vec{a} = -\omega^2 a_0 e^{-i\omega t_e} (\hat{x} + i\hat{y})$$



- On the x axis, $\vec{a}_T = -\omega^2 a_0 e^{-i\omega t_e} (i\hat{y})$, so $\vec{E}$ points on the y-axis
- On the y-axis, $\vec{a}_T = -\omega_0^2 a_0 e^{-i\omega t_e} (\hat{x})$ so the polarization points on the $\hat{x}$ axis
- On the z-axis $\vec{a}_T \propto e^{-i\omega t_e} (\hat{x} + i\hat{y})$ which is circular

$$\text{Re } \vec{a}_T = (\cos \omega t)\hat{x} + (\sin \omega t)\hat{y} = \overbrace{\text{right circular}}^{\text{like direction of motion}}$$

f) One can understand circular motion as a super-position of an x-oriented dipole and y-oriented dipole, 90° out of phase. On the x-axis only the radiation from the y-oriented dipole contributes as the x-oriented dipole is parallel to the observation direction (only transverse currents contribute to the radiation). On the z-axis both the x-oriented dipole + the y-oriented dipole contribute to the radiation field.



The two dipoles add incoherently.

Problem 5. A Charged Rotor: Zangwill

Two identical point charges of charge q are fixed to the ends of a rod of length 2ℓ which rotates with constant angular velocity of $\frac{1}{2}\omega$ in the $x-y$ plane about an axis perpendicular to the rod and through its center

(a) Calculate the electrid dipole moment $\boldsymbol{p}(t)$. Is there electric dipole radiation?

(b) Calculate the magnetic dipole moment $\boldsymbol{m}(t)$. Is there magnetic dipole radiation?

(c) Show that the electric quadruple moment is

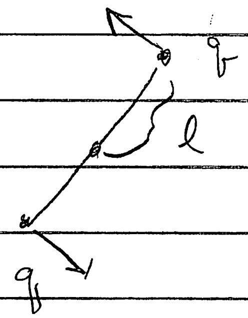
$$Q(t) = 3q\ell^2 \begin{pmatrix} \frac{1}{3} + \cos\omega t & \sin\omega t & 0 \\ \sin\omega t & \frac{1}{3} - \cos\omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \tag{25}$$

(d) Show that the time averaged angular distribution of radiated power is

$$\overline{\frac{dP}{d\Omega}} = \frac{1}{128\pi^2 c^5}\omega^6 q^2 \ell^4 (1 - \cos^4\theta)\,. \tag{26}$$

① Charged Rotor

Then consider



a) There is no electric dipole moment

b) There is a magnetic dipole moment

$$\vec{m} = \frac{1}{2c}\int d^3x \; \vec{x} \times \vec{J}$$

$$= \frac{1}{2c}\left[\omega q l^2 \hat{z} + \omega q l^2 \hat{z}\right] = \frac{\omega q l^2}{c}\hat{z}$$

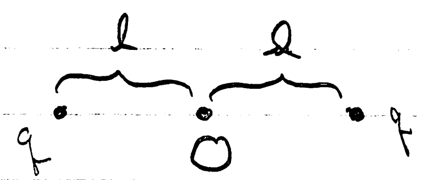
But the magnetic moment is constant and does not contribute to the radiation

② Charged Rotor

c) To find the quadrupole moment we note the definition

$$Q^{ij} = \int d^3\vec{r}\, \rho(\vec{r})\, (3r^i r^j - r^2 \delta^{ij})$$

Then lets find the Q at this time $(t=0)$ and then rotate



Q^{ij} is clearly zero for the off diagonal components. But the diagonal components are

$$Q^{xx} = \left[(3\ell^2 - \ell^2) + 3(-\ell)^2 - (-\ell^2)\right] q$$

$$Q^{xx} = 4q\ell^2$$

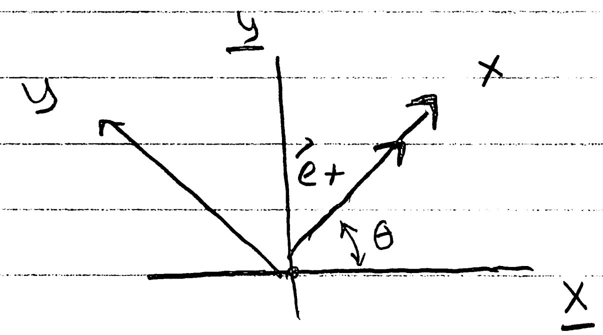
$$Q^{yy} = -2q\ell^2$$

$$Q^{zz} = -2q\ell^2$$

Now we know the components in the rotated coordinate system, We want the components in the unrotated system.

③ Charged Rotor

The appropriate rotation matrix is found from the picture:



$$\vec{e}_x = \cos\theta\, \underline{\vec{e}_x} + \sin\theta\, \underline{\vec{e}_y}$$

$$R^i{}_j = (\mathcal{R})^i{}_j = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

So

$$\underline{Q}^{ij} = (\mathcal{R})^i{}_l\, (\mathcal{R})^j{}_m\, Q^{lm}$$

Or in matrices

$$\underline{Q} = \mathcal{R}\, Q\, \mathcal{R}^T$$

$$\underline{Q} = q l^2 \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ \hline & & 1 \end{array}\right) \begin{pmatrix} 4 & & \\ & -2 & \\ & & -2 \end{pmatrix} \times$$

$$\left(\begin{array}{cc|c} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ \hline & & 1 \end{array}\right)$$

④ Charged Rotor

Multiplying out, & using

$$1 + 3\cos 2\theta = 4\cos^2\theta - 2\sin^2\theta,$$

gives with $2\theta = \omega t$

$$\underline{Q} = q\ell^2 \left(\begin{array}{cc|c} 1 + 3\cos\omega t & 3\sin\omega t & 0 \\ 3\sin\omega t & 1 - 3\cos\omega t & 0 \\ 0 & 0 & -2 \end{array}\right).$$

Or,

$$\underline{Q} = 3q\ell^2 \left(\begin{array}{cc|c} \frac{1}{3} + \cos\omega t & \sin\omega t & 0 \\ \sin\omega t & \frac{1}{3} - \cos\omega t & 0 \\ 0 & 0 & -2/3 \end{array}\right).$$

In terms of matrices:

$$(\star) \quad \underline{Q} = 3q\ell^2 \left[e^{-i\omega t}\sigma_z + i e^{-i\omega t}\sigma_x \right] + \text{const}$$

where

$$\text{Const} = \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & -2/3 \end{pmatrix} q\ell^2,$$

and σ_x and σ_z are pauli matrices. It is understood that we take the real part of Eq $\star$.

$$\sigma_x = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad \sigma_z \equiv \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

⑤ Charged Rotor

c) To find the power,

then we use

$$\vec{E}(t,\vec{r}) = -\frac{1}{24\pi r c^3}\left[\dddot{Q}\cdot\vec{n} - \vec{n}\left(\vec{n}^T\dddot{Q}\,\vec{n}\right)\right].$$

Looking at the form of $\overleftrightarrow{Q}$:

$$\overleftrightarrow{Q} = 3ql^2\left[e^{-i\omega t}\left(\sigma_z + i\sigma_x\right)\right] + \text{const}$$

we see that

$$\vec{E}(\vec{t},\vec{r}) = -i\vec{E}_1(t,r) + \vec{E}_2(t,r)$$

where $\vec{E}_1$ and $\vec{E}_2$ are real:

Eq. (*)

$$\begin{cases} \vec{E}_1(t,r) \equiv +\dfrac{3ql^2\omega^3}{24\pi r c^3}\left[(\sigma_z\cdot\vec{n}) - \vec{n}\left(\vec{n}^T\sigma_z n\right)\right] \\ \vec{E}_2(t,r) \equiv +\dfrac{3ql^2\omega^3}{24\pi r c^3}\left[(\vec{\sigma}_x\cdot\vec{n}) - \vec{n}\left(\vec{n}^T\sigma_x\vec{n}\right)\right] \end{cases}$$

So

$$\frac{\overline{dP}}{d\Omega} = \frac{c}{2}\,\mathrm{Re}\left[\vec{E}\cdot\vec{E}^*\right]r^2$$

$$\frac{dP}{d\Omega} = \frac{c}{2}\left(|E_1|^2 + |E_2|^2\right)r^2$$

⑥ Charged Rotor

So we evaluate the square of the bracketed terms $[\,]^2$ in Eq (*):

$$[\sigma \cdot \vec{n} - \vec{n}\,(n^T \sigma n)]^T \, [(\sigma \cdot n) - \vec{n}\,(n^T \sigma n)]$$

$$= (n^T \sigma - (n^T \sigma n)\, n^T)(\sigma \cdot n - n\,(n^T \sigma n))$$

$$= n^T \sigma^2 n - 2(n^T \sigma n)^2 + (n^T \sigma n)^2\, n^T n$$

$$= n^T \sigma^2 n - (n^T \sigma n)^2$$

We used that $\sigma^2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ for any pauli matrix and $n^T n = 1$ for a unit vector.

Then we find

$$(\bigstar)\quad \overline{\frac{dP}{d\Omega}} = \frac{c}{2}\left(\frac{3q\,\ell^2 \omega^3}{24\pi\, c^3}\right)^2 \left[2(n^T \sigma^2 n) - (n^T \sigma_x n)^2 - (n^T \sigma_z n)^2\right]$$

Taking $\vec{n}$ in the x-z plane:

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$

$$n^T \sigma_x n = (\sin\theta \;\; 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} = 0$$

$$n^T \sigma_z n = (\sin\theta \;\; 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} = \sin^2\theta$$

(7) Charged Rotor

and

$$n^T \sigma^2 n = (\sin\theta \;\; 0 \;\; \cos\theta) \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

$$= \sin^2\theta$$

So the term in square brackets in Eq ☆ on previous page is

$$[\,] = \left[2\sin^2\theta - (\sin^2\theta)^2 \right]$$

$$= \left[1 - \cos^4\theta \right]$$

And restoring constants we find from Eq ☆

$$\boxed{\frac{d\bar{P}}{d\Omega} = \frac{q^2 \ell^4 \omega^6}{128 \pi^2 c^5} (1 - \cos^4\theta)}$$

Problem 6. Basics of Relativity

(a) (Optional) The space time event at $X^\mu = (X^0, X^i) = (ct, \boldsymbol{x})$ happens at $\underline{X}^\mu = (\underline{X}^0, \underline{X}^i) = (c\underline{t}, \underline{\boldsymbol{x}})$ according to an observer moving to the right along the x axis with velocity v. Define the "light-cone" coordinates $x^+ \equiv X^0 + X^1$ and $x^- \equiv X^0 - X^1$. Show that under this boost that the x^+ coordinates are contracted, while the x^- coordinates are elongated

$$\underline{x}^+ = e^{-y} x^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, x^+ \,, \tag{27}$$

$$\underline{x}^- = e^{y} x^- = \sqrt{\frac{1+\beta}{1-\beta}}\, x^- \,. \tag{28}$$

Here

$$y = \tanh^{-1}\beta = \frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right) \tag{29}$$

is the so-called "rapidity" of the boost. What is $\underline{x}^+ \underline{x}^-$ and why is it unchanged under boost?

(b) (Optional) A Lorentz tensor transforms as

$$\underline{T}^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma T^{\rho\sigma} \tag{30}$$

Show that the transformation rule can be alternatively written

$$\underline{T}^\mu{}_\nu = (\mathcal{L})^\mu{}_\rho T^\rho{}_\sigma (\mathcal{L}^{-1})^\sigma{}_\nu \tag{31}$$

or equivalently

$$\underline{T}^\mu{}_\nu = L^\mu{}_\rho L_\nu{}^\sigma T^\rho{}_\sigma \tag{32}$$

(c) (Optional) The frequency and wave number of a plane wave of light, $e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{iK\cdot X}$, form a lightlike four vector

$$K^\mu = \left(\frac{\omega}{c}, \boldsymbol{k}\right) \tag{33}$$

(i) Show that $K \cdot K = K_\mu K^\mu = 0$ (this is the statement that K is lightlike.)

(ii) If a photon has frequency ω_o and is propagating along the z-axis, show (using the 4-vector properties of K^μ) that according to an observer propagating in the negative z direction with speeed β

$$\omega = \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_o \tag{34}$$

(d) (Optional) Show that the four velocity $U^\mu = dx^\mu/d\tau$ satisfies $U_\mu U^\mu = -c^2$.

(e) (Optional) For a particle with four momentum $P^\mu = (\frac{E}{c}, \boldsymbol{p}) = mU^\mu$ show that $P_\mu P^\mu = -(mc^2)^2/c^2$. This determines $E(\boldsymbol{p})$ the relation between energy and momentum:

$$\frac{E(\boldsymbol{p})}{c} = \sqrt{\boldsymbol{p}^2 + (mc)^2}\,. \tag{35}$$

(i) Show the velocity of the particle (i.e. the group velocity) is

$$\boldsymbol{v_p} \equiv \frac{\partial E(\boldsymbol{p})}{\partial \boldsymbol{p}} = \frac{c^2 \boldsymbol{p}}{E} \tag{36}$$

(f) (Do me! Not optional) A particle with velocity v_p in the x direction. Using the 4-vector transformation properties of U^μ, show that according to an observer moving to the right with velocity v, the particle moves with velocity

$$\underline{v}_p = \frac{v_p - v}{1 - v_p v/c^2} \tag{37}$$

<u>Basics</u> The lorentz transformation is

a)
$$\begin{pmatrix} \underline{X}^0 \\ \underline{X}^1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}$$

So

$$x^+ = X^0 + X^1$$

$$x^- = X^0 - X^1$$

And

$$\underline{x}^+ = \underline{X}^0 + \underline{X}^1$$

$$= \gamma X^0 - \gamma\beta X^1 + -\gamma\beta X^0 + \gamma X^1$$

$$= \gamma (X^0 + X^1) - \gamma\beta (X^0 + X^1)$$

$$= \gamma (1-\beta) \, x^+$$

$$\underline{x}^+ = \sqrt{\frac{1-\beta}{1+\beta}} \, x^+$$

Note

$$\tanh y = \beta \quad \text{or} \quad y = \frac{1}{2} \log \left(\frac{1+\beta}{1-\beta} \right)$$

So

$$\underline{x}^+ = e^{-y}\, x^+$$

Similarly

$$\underline{x}^- = \underline{x}^0 - \underline{x}^1$$

$$= \gamma(1+\beta)\, x^-$$

$$\underline{x}^- = \sqrt{\frac{1+\beta}{1-\beta}}\, x^-$$

$$\underline{x}^- = e^{+y}\, x^-$$

b) $$\underline{T}^{\mu\nu} = L^\mu{}_\rho\, L^\nu{}_\sigma\, T^{\rho\sigma}$$

Then raising/lowering σ and lowering ν

$$\underline{\underline{T}^\mu{}_\nu = L^\mu{}_\rho\, L_\nu{}^\sigma\, T^\rho{}_\sigma}$$

$$= (\mathcal{L})^\mu{}_\rho\, (\mathcal{L}^{-1\,T})_\nu{}^\sigma\, T^\rho{}_\sigma$$

$$\underline{\underline{T}^\mu{}_\nu = (\mathcal{L})^\mu{}_\rho\, T^\rho{}_\sigma\, (\mathcal{L}^{-1})^\sigma{}_\nu}$$

c) A plane wave of light is

$$e^{-i\frac{\omega}{c}X^0 + ikX^1} = e^{i\phi} \qquad (\star)$$

Under boost the speed of light is constaint. The only way to guarranty this is if the phase is invariant under Lorentz transformations

$$K^\mu = \left(\frac{\omega}{c}, k\right)$$

Then

$$K_\mu X^\mu = -\frac{\omega}{c} X^0 + k X^1$$

So

i) $$K_\mu K^\mu = -\left(\frac{\omega}{c}\right)^2 + k^2 = 0$$

since $\omega = ck$ is required in Eq $\star$

ii) Under boost in <u>negative</u> z-direction

$$\begin{pmatrix} \underline{\omega/c} \\ \underline{k} \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta \\ +\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \omega/c \\ k \end{pmatrix}$$

$$\underline{\omega}/c = \gamma\, \omega/c + \gamma\beta k$$

But $k = \omega/c$ so

$$\underline{\omega}/c = \gamma(1+\beta)\ \omega/c$$

$$\underline{\omega}/c = \sqrt{\frac{1+\beta}{1-\beta}}\ \omega/c$$

d) $U^\mu = \frac{dx^\mu}{d\tau}$ so

$$U_\mu U^\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{(dx_\mu dx^\mu)}{(d\tau)^2}$$

see below

$$= \frac{-c^2(d\tau)^2}{(d\tau)^2}$$

$$= -c^2$$

We used the definition of propper time $c^2 d\tau^2 = -dx_\mu dx^\mu = ds^2$

You can also use

$$U^\mu = (\gamma c, \gamma v) = c(\gamma, \gamma\beta)$$

So

$$U_\mu U^\mu = c^2\left[-\gamma^2 + \gamma^2\beta^2\right] = -c^2$$

Problem 7. One liners

(a) Starting from the Maxwell equations for $F^{\mu\nu}$ and the definition of $F^{\mu\nu}$, derive the wave equation $-\Box A^\mu = J^\mu/c$.

(b) Starting from the maxwell equations for $F^{\mu\nu}$ in covariant form, show that we must have $\partial_\mu J^\mu = 0$ for consistency.

(c) (This is two lines) Show that the energy conservation and force laws

$$\frac{dE_{\boldsymbol{p}}}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}_p \tag{38}$$

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \frac{\boldsymbol{v_p}}{c}\times\boldsymbol{B}) \tag{39}$$

can be written covariantly

$$\frac{dP^\mu}{d\tau} = F^{\mu\nu}u_\nu/c \tag{40}$$

Note that $E_{\boldsymbol{p}}$ (the energy of the particle) is different from $\boldsymbol{E}$ the electric field.

(d) From Eq. (40) show that $P_\mu P^\mu$ is constant in time.

(e) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transform

$$A_\mu \to A_\mu + \partial_\mu \Lambda(X) \tag{41}$$

where Λ is an arbitrary function of $X = (t, \boldsymbol{r})$.

(f) Given $F^{\mu\nu}$ the only two Lorentz invariant quantities are $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Evaluate these two invariants in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$[1]

[1] answers: $2(B^2 - E^2)$ and $-4E\cdot B$

One liners

a) $$-\partial_\mu F^{\mu\nu} = J^\nu/c$$

$$-\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\nu/c$$

$$-(\partial_\mu \partial^\mu) A^\nu + \partial^\nu \cancel{(\partial_\mu A^\mu)}^{0} = J^\nu/c$$

with $\partial_\mu A^\mu = 0$ and $\Box = \partial_\mu \partial^\mu$, find

$$-\Box A^\nu = J^\nu/c$$

b) $$\partial_\mu F^{\mu\nu} = J^\nu/c$$

$$\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{\text{this}} = \partial_\nu J^\nu/c$$

$F^{\mu\nu}$ is anti-symmetric, while $\partial_\mu \partial_\nu$ is symmetric, so this is zero

$$\partial_\nu J^\nu/c = 0$$

c) $$\frac{dE_p}{dt} = q\,\vec{E} \cdot \vec{v}_p$$

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}_p}{c} \times \vec{B}\right)$$

Then multiply both sides by γ. Use

$$\gamma \frac{d}{dt} = \frac{d}{d\tau} \qquad \frac{dt}{\gamma_p} = d\tau$$

So that with $u^\mu = (\gamma c, \gamma v)$ and $E^i = F^{0i}$,

$$\frac{d(E_p/c)}{d\tau} = q F^{0i} \frac{u_i}{c} \qquad F_{ij} = \varepsilon_{ijk} B^k,$$

$$\frac{dp^i}{d\tau} = q\left(E^i \gamma + \varepsilon^i{}_{jk} \gamma \frac{v_p^j}{c} B^k\right)$$

$$= q\left(F^i{}_0 \frac{u^0}{c} + F^i{}_j \frac{u^j}{c}\right)$$

So

$$\underline{\underline{\frac{dp^\mu}{d\tau} = q F^\mu{}_\nu \frac{u^\nu}{c}}}$$

d) $$p^\mu \frac{dp_\mu}{d\tau} = q F_{\mu\nu} \frac{p^\mu u^\nu}{c} = \frac{qm}{c} F_{\mu\nu} u^\mu u^\nu$$

$$= 0$$

since $F_{\mu\nu}$ is antisymm while

$u^\mu u^\nu$ is symmetric.

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e) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

So $A_\mu \rightarrow \underline{A}_\mu = A_\mu - \partial_\mu \Lambda$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \overbrace{- \partial_\mu \partial_\nu \Lambda + \partial_\nu \partial_\mu \Lambda}^{=0}$$

f) $F_{\mu\nu} F_\nu^{\mu\nu} = -2E^2 + 2B^2 = -2(E^2 - B^2)$

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & \vec{E} & \\ \hline -\vec{E} & 0 & B_z & -B_y \\ & & 0 & B_x \\ & -\vec{B} & & 0 \end{array}\right)$$

$$F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & -\vec{E} & \\ \hline E & 0 & B^z & -B^y \\ & & 0 & B^x \\ & -B & & 0 \end{array}\right)$$

$$\tilde{F}_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & \vec{B} & \\ \hline -\vec{B} & 0 & E^z & -E^y \\ & -E^z & 0 & E^x \\ & E^y & -E^x & 0 \end{array}\right)$$

$F_{\mu\nu} F^{\mu\nu} = -4 \vec{E} \cdot \vec{B}$