Magnetic Matter

determined with electrostatics

- $$\nabla \times \vec{B} = \frac{\vec{j}}{c} + \frac{1}{c}\frac{\partial \vec{E}^{(0)}}{\partial t}$$

$$\nabla \cdot B = 0$$

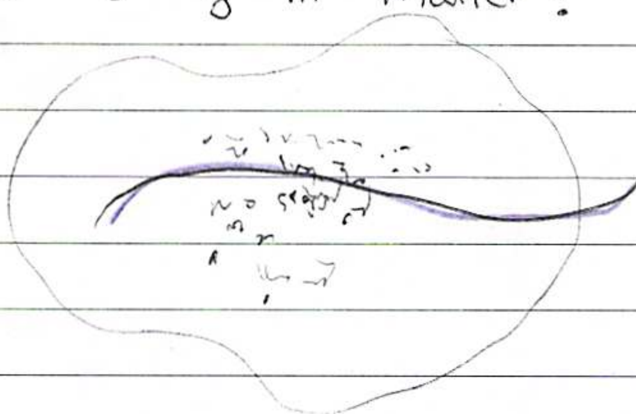
- For the moment set $E^{(0)} = 0$ (no net charge)

$$\nabla \times \vec{B} = \frac{j}{c} = \frac{j_{mat}}{c} + \frac{j_{free}}{c}$$

or external

$$\nabla \cdot B = 0$$

What is $\vec{j}$ in matter?



long wavelength magnetic field

$L \gg l_{micro}$

Basic Principles

- Symmetry and long distance expansion
- very often weak

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- Consider a magnetic field slowly varying in time and space. Write $\vec{j}$ as some general fcn of $\vec{B}$ and its derivatives

$$\vec{j} = \sigma_B \vec{B} + \chi_1 \partial_t \vec{B} + \chi_m (\nabla \times \vec{B}) + \ldots$$

+ higher derivatives
as before these are suppressed by powers of $\frac{l_{micro}}{L}$

- Now recognize that symmetry forces many of these to be zero. Take the first term,

$$\vec{j} = \sigma_B \vec{B}$$

$\vec{j}$: P-odd, T-odd

σ_B: P-odd, T-even

$\vec{B}$: P-even, T-odd

- The coefficients σ_B reflect the microscopic interactions. If the microscopic forces are invariant under parity.

$$\vec{x} \longrightarrow -\vec{x}$$

Then σ_B will be zero $\sigma_B = 0$

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(But in other cases such as high temperatue electroweak plasma, which violates parity, this coefficient will not be zero. It will not be dissipative)

- Unless otherwise specified we will assume parity invariance of microscopic forces and set $\sigma_B = 0$

$$\vec{j} = \cancel{\sigma_B \vec{B}} + \underbrace{\cancel{x_1 \partial_t \vec{B}}}_{\text{P-odd}} + \chi_m^B c(\nabla \times \vec{B}) + \ldots$$

So $\boxed{\frac{\vec{j}}{c} = \chi_m^B \vec{\nabla} \times \vec{B}} \implies \boxed{\frac{j}{c} = \nabla \times (\underbrace{\chi_m^B \vec{B}}_{\equiv \vec{M} = \text{magnetization}})}$

- Now our equations of magneto-statics becomes

$$\nabla \times B = j_{mat} + \frac{j_{free}}{c} \qquad \nabla \cdot B = 0$$

$$\nabla \times \vec{B} = \nabla \times \vec{m} + \frac{j_{free}}{c}$$

And

$$\nabla \times \underbrace{(\vec{B} - \vec{m})}_{\equiv \vec{H}} = \frac{j_{free}}{c}$$

Leading to our eqs - Magneto Statics in matter

$$\boxed{\begin{aligned} \nabla \times \vec{H} &= \frac{\vec{j}_{fr}}{c} \\ \nabla \cdot \vec{B} &= 0 \end{aligned}}$$

$H[\vec{B}] = \vec{B} - \vec{M}(\vec{B})$

$$\boxed{\vec{\nabla} \times \vec{M} = \vec{j}_{mat}}$$

For a linear relation $\vec{M} = \chi^m_B \vec{B}$ and

$$\vec{H} = (1 - \chi^B_m)\vec{B}$$

$$\frac{1}{(1-\chi^B_m)} \vec{H} = \vec{B}$$

$$\boxed{\mu \vec{H} = \vec{B}}$$

$$\boxed{\mu = \frac{1}{1-\chi^B_m} = \text{permeability}}$$

and we get a set of equations; with $\vec{B} = \nabla \times \vec{A}$, for constant μ

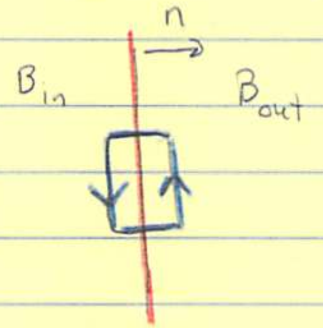
$$\nabla \times \left(\frac{\nabla \times A}{\mu}\right) = j_{free}/c$$

$$\boxed{-\nabla^2 A = \mu\, j_{free}/c}$$

This assumes a linear relation

Boundary Condititions; previously found

(☆)

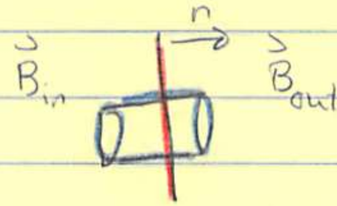


$\nabla \times B = j/c$

$\Longrightarrow$

$\vec{n} \times (\vec{B}_{out} - \vec{B}_{in}) = \frac{\vec{K}_{TOT}}{c}$

$\nabla \cdot B = 0$



$\vec{n} \cdot (\vec{B}_{out} - \vec{B}_{in}) = 0$

In Matter:

$\nabla \times \vec{H} = j_{ext}/c$

$\nabla \cdot B = 0$

$\Longrightarrow$

$$\boxed{\begin{aligned} n \times (\vec{H}_{out} - \vec{H}_{in}) &= \frac{\vec{K}_{ext}}{c} \\ n \cdot (\vec{B}_{out} - B_{in}) &= 0 \end{aligned}}$$ (☆☆)

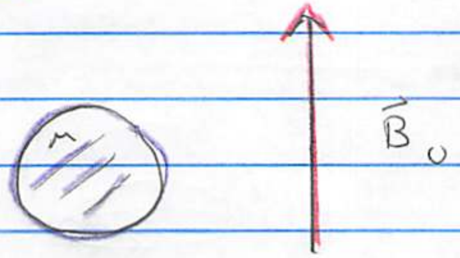
Using $\vec{H} = \vec{B} - \vec{m}$ and $\vec{K}_{TOT} = \vec{K}_{mat} + \vec{K}_{ext}$
in Eq (☆) and Eq (☆☆):

Find

$$\boxed{\vec{n} \times (\vec{m}_{out} - \vec{m}_{in}) = \vec{K}_{mat}/c}$$

in words the jump in the magnetization determines the current.

Example



Magnetizable Sphere in a constant magnetic field

- The magnetic field tends to align the magnetic dipoles making up the material inducing a dipole moment

$$\nabla \times H = \cancel{j_{ext}}^{\,0}$$

$$\nabla \cdot B = 0 \quad \Rightarrow \quad B = \nabla \times \vec{A} \quad \text{then} \quad -\nabla^2 \vec{A} = \cancel{\mu j_{ext}}^{\,0}$$

- $\vec{B}$ is produced by currents in the azimuthal direction. Try $\vec{A} = A_\phi(r,\theta)\,\hat{\phi}$. Writing out the vector laplacian in these coordinates we have

$$-\nabla^2 \vec{A} = \left(-\nabla^2 A_\phi + \frac{A_\phi}{r^2 \sin^2\theta} \right) \hat{\phi} = 0$$

- Seporation of variables $A = R(r)\, P(\theta)$ gives

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] R(r) = 0$$

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \right] P(\theta) = \ell(\ell+1)\, P(\theta)$$

new bit

- The only difference from before is $1/\sin^2\theta$ which produces a new set of eigenfunctions

$$P_\ell^1(\cos\theta) = P_\ell^m(\cos\theta) \quad \text{with } m=1$$

- The radial equation is the same

$$A_\phi(r,\theta) = \sum_\ell \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right) P_\ell^1(\cos\theta)$$

- A sample of associated Legendre Polynoms is given on the next page

- The boundary data is given by the magnetic field as $r \to \infty$

$$\vec{A} = \frac{1}{2}\vec{B}_0 \times \vec{r} \qquad \text{(See Previous Lecture)}$$

So

$$A_\phi = \frac{1}{2} B_0 \underbrace{\sin\theta}_{P_1^1(\cos\theta)}\, r$$

- Thus the boundary data only involves $\ell = 1$ and we are motivated to try a solution involving $\ell = 1$

$$A_\phi = \left(C r + \frac{D}{r^2}\right)\sin\theta$$

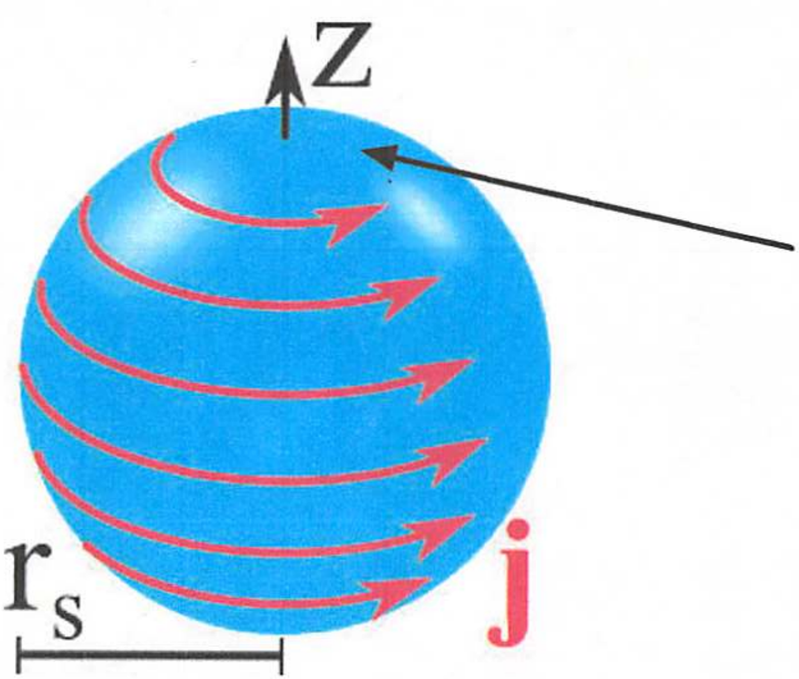
try this form inside & outside

The associated legendre polynomial behave as

$$P_\ell^1(\cos\theta) \propto \sin(\theta) \times (\text{polynomial in } \cos\theta)$$

so that the current is regular at the top and bottom of the sphere.

$$P_1^1(\cos\theta) = \sin\theta \tag{1}$$
$$P_2^1(\cos\theta) = -\,3\sin\theta\cos\theta \tag{2}$$
$$P_3^1(\cos\theta) = -\,\frac{3}{2}\sin\theta\,(5\cos^2\theta - 1) \tag{3}$$
$$P_4^1(\cos\theta) = \ldots \tag{4}$$



Must be regular here, $j_\phi \propto \sin\theta$

- Outside with $r \to \infty$ requirement gives

$$A_\phi^{out} = \left(\frac{1}{2} B_0 r + \frac{D^{out}}{r^2}\right) \sin\theta$$

- Inside with regularity requirement gives

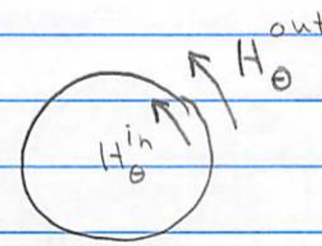
$$A_\phi^{in} = C^{in}\, r \sin\theta$$

Demanding continuity at $r = a$

$$\frac{1}{2} B_0 a + \frac{D^{out}}{a^2} = C^{in} a \qquad (1)$$

- Now we need to compute $\vec{B}$ and $\vec{H}$

$$B_\theta = (\vec{\nabla} \times \vec{A})_\theta = -\partial_r A_\phi$$



$$H_\theta^{in} = \mu B_\theta^{in} \qquad H_\theta^{out} = B_\theta^{out}$$

So $$\boxed{H_\theta^{out} - H_\theta^{in} = \overset{0}{\cancel{K_\theta^{ext}}}/c}$$ ← Our boundary conditions

$$\left(-\frac{1}{2} B_0 + \frac{2D^{out}}{a^3}\right) - C^{in} = 0 \qquad (2)$$

Soving the two equations (1) + (2) for D^{out} and C^{in}

L

We find

$$D^{out} = B_0 a^3 \left(\frac{\mu-1}{\mu+2}\right) \qquad C^{in} = \frac{3\mu}{2+\mu} \frac{B_0}{2}$$

So

$$\boxed{\begin{aligned} A_\phi^{out} &= \left(\frac{1}{2} B_0 r + \frac{B_0 a^3}{r^2}\left(\frac{\mu-1}{\mu+2}\right)\right) \sin\theta \qquad r > a \\ A_\phi^{in} &= \frac{3\mu}{\mu+2} \frac{B_0}{2} r \sin\theta \qquad r < a \end{aligned}}$$

Outside, this is again the field of a dipole

$$\vec{B}_{out} = B_0 \hat{z} + \frac{3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}}{4\pi r^3} \qquad r > a$$

where $\vec{m} = 4\pi B_0 a^3 \left(\frac{\mu-1}{\mu+2}\right)$

Inside, this is again a constant field

$$\vec{B}_{in} = \frac{3\mu}{\mu+2} \vec{B}_0 \qquad \vec{H} = \frac{\vec{B}}{\mu}$$

Check our Solution

Now lets check that Boundary Conditions Are Satisfied

- The surface current is

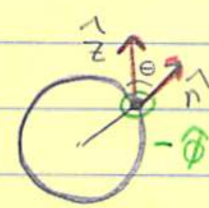
$$\vec{n} \times (\vec{m}_{out} - \vec{m}_{in}) = \vec{K}_{mat}/c$$

- Now the magnetization outside the sphere is $= 0$ since we have no medium outside. Thus

$$-\vec{n} \times \vec{m}_{in} = \vec{K}_{mat}/c$$

- Then $\vec{m} = (\mu - 1)\vec{H}$, so

$$\frac{\vec{K}_{mat}}{c} = -\vec{n} \times \left[(\mu - 1) \frac{3B_0}{2+\mu} \hat{z} \right]$$



$$\frac{\vec{K}_{mat}}{c} = (\mu - 1) \frac{3B_0}{2+\mu} \underbrace{(-\vec{n} \times \hat{z})}_{-\hat{\phi}}$$

see picture

$$\boxed{\frac{\vec{K}_{mat}}{c} = 3B_0 \left(\frac{\mu - 1}{\mu + 2} \right) \sin\theta \, \hat{\phi}}$$

- Thus, we see that the current distribution on the surface of the sphere:

$$\vec{K} \propto \sin\theta \, \hat{\phi} \,,$$

is the same as for the rotating charged sphere, and thus the the induced magnetic fields in this case are the same (up to constant) as for the rotating charged sphere.