Steady Currents and Ohms Law

- Given a set of potentials how do I calculate the current flow?

- Well ... you specify the currents and solve for the fields

$$\vec{j} = \sigma \vec{E} \quad \leftarrow \text{now solve for fields}$$

- Then if the current is steady

$$\cancel{\partial_t \rho}^{\,0} + \nabla \cdot \vec{j} = 0$$

or

$$\nabla \cdot (\sigma \vec{E}) = 0$$

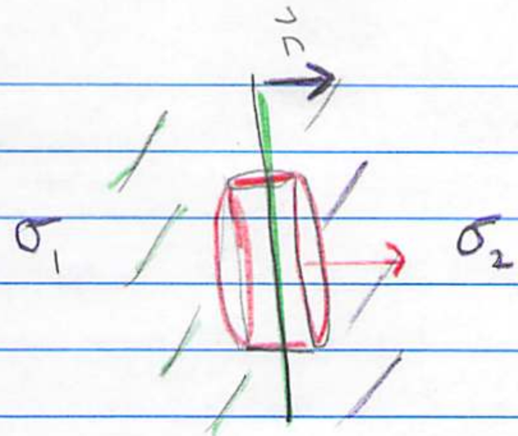
and

$$\nabla \times \vec{E} = 0 \quad \leftarrow \text{so } \vec{E} = -\nabla \varphi$$

- Thus we find an equation to solve $\vec{E} = -\nabla \varphi$:

$$\nabla \cdot (\sigma \vec{\nabla} \varphi) = 0$$

This for constant σ is the same old Laplace Eqn, but the boundary conditions are different, and this makes the solutions different



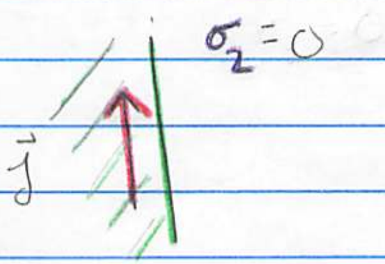
- From $\nabla \cdot \vec{j} = 0$, we integrate over the box

$$\int dV \, \nabla \cdot \vec{j} = A \, \vec{n} \cdot (\vec{j}_2 - \vec{j}_1) = 0$$

- So we find

$$\vec{n} \cdot (\vec{j}_2 - \vec{j}_1) = 0 \implies \boxed{\sigma_2 E_2^{\perp} - \sigma_1 E_1^{\perp} = 0}$$

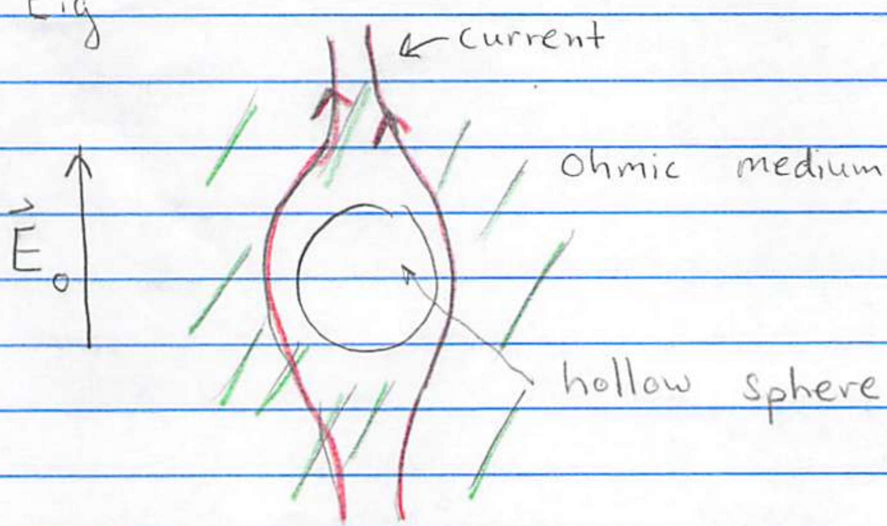
- This is most often used at an interface where $\sigma_2 = 0$. Then we find the b.c.



$$E_1^{\perp} = -\vec{n} \cdot \nabla \varphi = 0$$

This means the normal derivative is zero and is a Neumann type boundary condition

E.g



- Let us solve for the current

Solving

$$-\sigma \nabla^2 \varphi = 0$$

$$\varphi = \sum_{\ell} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}}\right) P_{\ell}(\cos\theta)$$

Try this ansatz from experience with similar problems

$$\varphi = -E_0 r\cos\theta + \frac{B}{r^2}\cos\theta$$

- Now we want $-\vec{n}\cdot\nabla\varphi = 0 \Rightarrow -\left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = 0$ on surface

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = -E_0\cos\theta - \frac{2B}{a^3}\cos\theta$$

or $$B = -\frac{E_0 a^3}{2}$$

- Thus

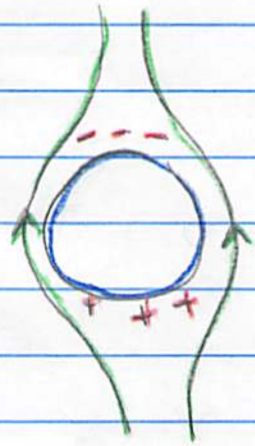
$$\varphi_{out} = -E_0\left(r + \frac{a^3}{2r^2}\right)\cos\theta = -E_0 z - \frac{a^3}{2r^2}E_0\cos\theta$$

And

$$\vec{J} = \sigma\vec{E} = \sigma(-\nabla\varphi)$$

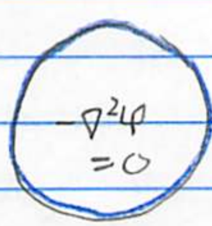
$$\vec{J} = \sigma E_0 \hat{z} \underbrace{- \sigma E_0 \frac{a^3}{r^3} \cos\theta \, \hat{r}}_{\text{from } \frac{\partial \varphi}{\partial r}\hat{r}} \underbrace{- \sigma E_0 \frac{a^3}{2r^3} \sin\theta \, \hat{\theta}}_{\text{from } \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\theta}}$$

- Which has a fluid flow character to it



- Note that the current is deflected because surface charges build up on the sphere wall

- To determine these charges we need to solve for the potential inside the sphere. On the boundary we have a continuous potential



$\varphi_0 = \varphi$ on boundary, i.e. $r = a$

$$\varphi_0 = -\frac{3}{2} a E_0 \cos\theta$$

- You can check that

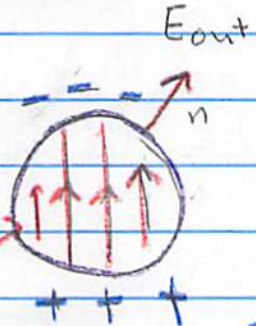
$$\varphi_{in} = -\frac{3}{2} r E_0 \cos\theta = \underbrace{-\frac{3}{2} E_0 z}_{\text{this is a constant field in the } z \text{ direction}}$$

$$\vec{E} = \frac{3}{2}\vec{E}_0$$

solves the boundary condition and $-\nabla^2\varphi = 0$.

- The general boundary condition $\vec{n}\cdot(\vec{E}_2-\vec{E}_1)=\sigma$ gives the charge

$$\overset{0}{\vec{n}\cdot\vec{E}_{out}} - \vec{n}\cdot\vec{E}_{in} = \sigma$$



So

$$\frac{\partial\varphi_{in}}{\partial r} = \sigma$$

$$-\frac{3}{2}E_0\cos\theta = \sigma$$

this agrees with intuition: + on bottom − on top.

Electric Field inside

surface charge

the Electric field outside is // to the surface so $\vec{n}\cdot\vec{E}_{out} = 0$. If $E_{\perp,out}$ were non-zero current would flow into the sphere