

5 Ohms Law and Conduction

5.1 Steady current and Ohms Law:

- (a) For steady currents

$$\nabla \cdot \mathbf{j} = 0 \quad (5.1)$$

- (b) For steady currents in ohmic matter

$$\mathbf{j} = \sigma \mathbf{E} \quad (5.2)$$

- (c) σ has units of $1/s$. Note that in MKS units σ_{MKS} has the uninformative unit $1/\text{ohm m}$:

$$\sigma_{HL} = \frac{\sigma_{MKS}}{\varepsilon_o} \quad (5.3)$$

For $\sigma_{MKS} = 10^7 (\text{ohm m})^{-1}$ we find $\sigma \sim 10^{18} \text{s}^{-1}$.

- (d) To find the flow of current we need to solve the electrostatics problem

$$-\nabla \cdot (\sigma \mathbf{E}) = 0 \quad (5.4)$$

$$\nabla \times \mathbf{E} = 0 \quad (5.5)$$

or for homogeneous material ($\sigma = \text{const}$)

$$-\sigma \nabla^2 \Phi = 0 \quad (5.6)$$

We see that we are supposed to solve the Laplace equation. However the boundary conditions are rather different.

- (e) A point source of current is represented by a delta function $I\delta^3(\mathbf{r} - \mathbf{r}_o)$. While a sink of current is represented by a delta function of opposite sign $-I\delta^3(\mathbf{r} - \mathbf{r}_o)$.
- (f) Eq. (5.4) and Eq. (5.6) need boundary conditions. At an interface current should be conserved so

$$\mathbf{n} \cdot (\mathbf{j}_2 - \mathbf{j}_1) = 0 \quad (5.7)$$

or

$$\sigma_2 \frac{\partial \Phi_2}{\partial n} = \sigma_1 \frac{\partial \Phi_1}{\partial n} \quad (5.8)$$

Most often this is used to say that the normal component of the Electric field at a metal-insulator interface should be zero:

$$\mathbf{n} \cdot \mathbf{E} = 0 \quad \text{at metal-insulator interface} \quad (5.9)$$

- (g) In general the input current (or normal derivatives of the potential) must be specified at all the boundaries in order to have a well posed boundary value problem that can be solved (at least numerically.)
- (h) In general the input currents $I_a = I_1, I_2, \dots$ on a set conductors will be specified, specifying the normal derivatives on all of the surfaces. Then you solve for the potential. The voltages of a given electrode relative to ground is V_a , and you will find that $V_a = \sum_b R_{ab} I_b$. R_{ab} is the resistance matrix.

5.2 Basic physics of metals, Drude model of conductivity:

This section really lies outside of electrodynamics. But it helps to understand what is going on.

- (a) The electrons in the metal under go scatterings with impurities and other defects on a time scale τ_c .
For copper:

$$\tau_c \sim 10^{-14} \text{s} \quad (5.10)$$

- (b) A typical coulomb oscillation / orbital frequency is set by the plasma frequency

$$\omega_p = \sqrt{\frac{ne^2}{m}} \quad (5.11)$$

For copper ω_p is of order a typical quantum frequency and scales like:

$$\omega_p \sim \left(\frac{1}{m} \underbrace{\frac{e^2}{a_o^3 m}}_{\text{spring const}} \right)^{1/2} \quad (5.12)$$

$$\sim \left(\frac{27.2 \text{ eV}}{\hbar} \right) \quad (5.13)$$

$$\sim 10^{-16} \text{ 1/s} \quad (5.14)$$

In the second to last line we ignored all 4π factors and used Bohr model identities

$$\frac{1}{2} \left(\frac{e^2}{4\pi a_o} \right) = \frac{\hbar^2}{2ma_o^2} = 13.6 \text{ eV} \quad (5.15)$$

which you can remember by noting that (minus) coulomb potential energy is twice the kinetic energy $= p^2/2m$ and knowing $p_{\text{bohr}} = \hbar/a_o$ as expected by the uncertainty principle.

- (c) Since the distances between collisions are long compared to the Debroglie wavelength, and the time between collisions is long compared to a typical inverse quantum frequency, we are justified in using classical transport

$$\omega_p \tau_c \sim 100 \gg 1 \quad (5.16)$$

- (d) In the Drude model the magnitude of the driving force $F_E = eE_{\text{ext}}$ equals the magnitude drag force $F_{\text{drag}} = m\mathbf{v}/\tau_c$, leading to an estimate of the conductivity

$$\sigma = \frac{ne^2\tau_c}{m} = \omega_p^2 \tau_c \quad (5.17)$$

The estimates given show

$$\sigma \sim 10^{18} \text{ s}^{-1} \quad (5.18)$$

for a metal like copper.