# Self Adjoint Linear Ops and Eigenvalue Problems

- Take the inner product on $(a,b)$

$$\langle f,g\rangle = \int_a^b w(x)\, f^*(x)\, g(x)$$

We require that $w(x) > 0$ and $p(x) \neq 0$ in the interval $(a,b)$

- With the right b.c. the Sturm-Liouiville Op is self adjoint $w(x) > 0$ $p(x), q(x)$ real

$$\mathcal{L}_x \equiv \frac{1}{w(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right]$$

$\mathcal{L}_x$ is self adjoint provided certain boundary conditions are met. Take two functions $\phi_1$ and $\phi_2$. We want to show that $\langle \phi_1, \mathcal{L}_x\phi_2\rangle = \langle \mathcal{L}_x\phi_1, \phi_2\rangle$

- Just integrate twice by parts (Do it!)

$$\int_a^b w(x)(\mathcal{L}_x\phi_1)^*\phi_2 = -p(x)\left(\frac{d\phi_1^*}{dx}\phi_2 - \phi_1^*\frac{d\phi_2}{dx}\right)\Bigg|_a^b$$

$$+ \int_a^b dx\, w(x)\, \phi_1^*(x)(\mathcal{L}_x\phi_2)$$

So the operator is self adjoint if

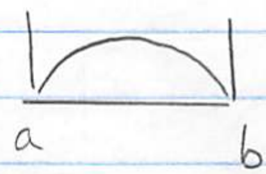
$$-p(x)\left(\frac{d\phi_1^*}{dx}\phi_2 - \phi_1^*\frac{d\phi_2}{dx}\right)\Bigg|_a^b = 0$$

Bndry Terms ← this is known as the bilinear concomitant, i.e. flux

# Examples of B.C. Where Boundary Terms Vanish

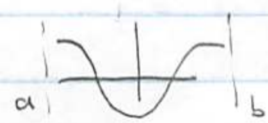
- Particle in box, Dirichlet B.C.

$$\left(\frac{d^2}{dx^2} + k^2\right)\phi = 0 \qquad \phi(a) = \phi(b) = 0$$




With these b.c.
the boundary terms vanish
a the operator is self adjoint

- Neumann b.c. $\phi'(a) = \phi'(b) = 0$ also leads to a self-adjoint op.




- In general a two-point homogeneous b.c. of the form

$$\alpha\phi + \beta\phi' = 0 \quad \text{at} \quad x = a \ \underline{\text{and}} \ x = b$$

leads to a self-adjoint op.

- Regularity at singular points.
Often a or b will correspond to zero's of $p(x)$ (which is a singular point of the DEQ). Demanding regularity at this point, we have for instance

$$\lim_{x \to a} p(x)\left(\frac{d\phi_1^*}{dx}\phi_2 - \phi_1^*\frac{d\phi_2}{dx}\right) = 0 \qquad p(a) = 0$$

$\phi$ regular

# Eigen-fcns of Sturm Liouville operators

- We Look for solutions of the form

This is a kind of generalized schrd̈inger equation

$$\frac{1}{w(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx}+q(x)\right]\phi_n=\lambda_n\phi_n$$

or $$\left[-\frac{d}{dx}p(x)\frac{d}{dx}+q(x)\right]\phi_n=\lambda_n w(x)\,\phi_n$$

with self-adjoint b.c.

The eigen functions are orthogonal:

(1) $$\langle\phi_{n_1},\phi_{n_2}\rangle=\int_a^b dx\; w(x)\,\phi^*_{n_1}\phi_{n_2}=\begin{cases}0 & \lambda_1\neq\lambda_2\\ C_n & \lambda_1=\lambda_2\end{cases}$$

And complete

(2) $$\sum_n\frac{\phi(x)\,\phi^*(x')}{C_n}=\frac{1}{W(x)}\,\delta(x-x')$$

Any function satisfying the b.c. can be expanded

- $$f(x)=\sum_{x} f_n\frac{\phi_n}{C_n}$$

- $$f_n=\int_a^b dx\; W(x)\,\phi^*_n(x)\,f(x)=\langle\phi_n,f\rangle$$

## Consistency of ②

$$f(x) = \sum_n \underbrace{\int_a^b dx'\, w(x')\, \phi_n^*(x')\, f(x')}_{f_n} \; \frac{\phi_n(x)}{c_n}$$

$$= \int_a^b dx'\, w(x') \sum_n \frac{\phi_n(x)\, \phi_n^*(x')}{c_n} \; f(x')$$

This needs to hold for all $f(x)$, and so we must have

$$= \int_a^b dx' \; w(x') \; \frac{1}{w(x')} \; \delta(x - x') \; f(x') = f(x)$$

- Then Proof of ①. For <u>any</u> two functions $\phi_1$ and $\phi_2$ we showed

$$\int_a^b dx\, w(x) (\mathcal{L}_x \phi_1)^* \phi_2 = -p(x) \left( \frac{d\phi_1^*}{dx} \phi_2 - \phi_1^* \frac{d\phi_2}{dx} \right) \Bigg|_a^b + \int_a^b dx\, w(x)\, \phi_1^* (\mathcal{L}_x \phi_2)$$

- Now if $\phi_1$ and $\phi_2$ are eigen-fcns obeying the b.c. then the boundary terms vanish. And $\mathcal{L}_x \phi_n = \lambda_n \phi_n$ yielding (note $\mathcal{L}_x(\phi^*) = (\mathcal{L}_x \phi)^* = \lambda\, \phi^*$

$$(\lambda_1 - \lambda_2) \int_a^b dx\, w(x)\, \phi_1^*(x)\, \phi_2(x) = 0$$

So provided $\lambda_1 \neq \lambda_2$ we have

$$\langle \phi_1, \phi_2 \rangle = 0$$

Further analysis shows that there is no degeneracy in one dimension