Schwinger Keldysh

Consider Computing in HO:

$$G^{>}(t,\bar{t}) = \langle \hat{x}(t)\,\hat{x}(\bar{t}) \rangle$$ ↗ not $\langle T[\hat{x}\hat{x}]\rangle$

↑ not useful for T ≠ 0

Ok

$$G^{>}(t,\bar{t}) = \mathrm{Tr}\,\rho\,\hat{x}(t)\,\hat{x}(\bar{t})$$

$$G^{>}(t,\bar{t}) = \int dx_1^\circ \int dx_2^\circ \;\rho(x_2^\circ, x_1^\circ)\;\langle x_2^\circ|\,\hat{x}(t)\,x(\bar{t})\,|x_1^\circ\rangle$$

So

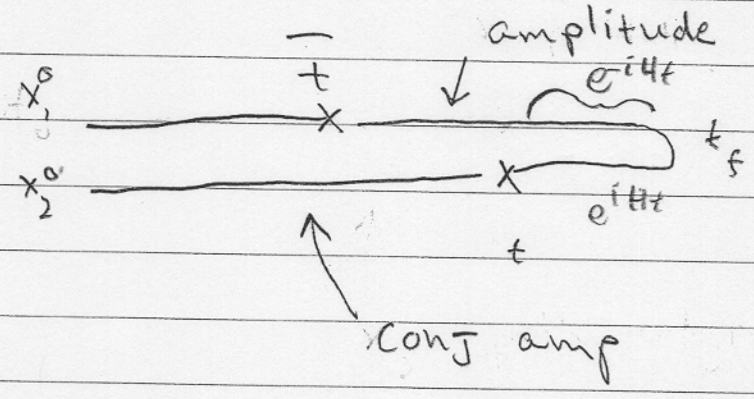
$$\int dx_f \;\underbrace{\langle x_2^\circ|\,\hat{x}(t)\,|x_f\rangle}_{\text{Conj-amp @ insertion}}\underbrace{\langle x_f|\,\hat{x}(\bar{t})\,|x_1^\circ\rangle}_{\text{Amplitude with insertion at }\bar{t}}$$

$$\text{Amp} = \int_{x_1^\circ}^{x_f} Dx_1\, e^{iS_1}\, x_1(\bar{t})$$

$$\text{Conj amp} = \left[\langle x_f|\,\hat{x}(t)\,|x_2^\circ\rangle\right]^* = \int_{x_2^\circ}^{x_f} Dx_2\; e^{-iS_2}\; x_2(t)$$

So

$$G^{>}(t,\bar{t}) = \int dx_1^0 dx_2^0 \, \rho(x_1^0 x_2^0) \int_{x_1^0 x_2^0} Dx_1 Dx_2 \, e^{iS_1 - iS_2} \; X_2(t)\, x_1(\bar{t})$$



$$x_1 = x_2 \Big|_{t = t_f}$$

Now

$G_{21} \equiv G^{>}(t,\bar{t}) = \langle x_2(t)\, x_1(\bar{t}) \rangle$

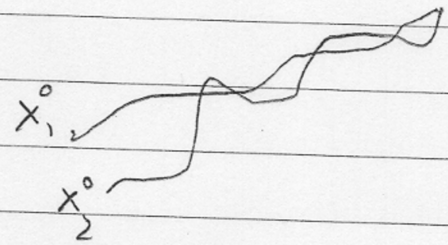
$G_{12} \equiv G^{<}(t,\bar{t}) = \langle x_1(t)\, x_2(\bar{t}) \rangle$

Not useful for $T \neq 0$

$\langle T[\hat{x}\, \hat{x}] \rangle = \langle x_1 x_1 \rangle$

$\langle \tilde{T}[\hat{x}\ \hat{x}] \rangle = \langle x_2 x_2 \rangle$

Now think about semi-classical limit



Amplitude is close to conjugate amp

$$x_r = \frac{x_1 + x_2}{2} \qquad x_a = x_1 - x_2 \Rightarrow \text{small}$$

$$x_a \Big|_{t_f} = 0$$

$$x_1 = x_r + x_a/2$$

Now, this is useful

$$Z = \int Dx_r \, Dx_a \, e^{iS[x_1] - iS[x_2]}$$

small ↓

$$iS[x_1] - iS[x_2] = iS[x_r + x_a/2] - iS(x_r - x_a/2]$$

$$\simeq \int_t i \, x_a(t) \frac{\delta S}{\delta x}\Big|_{x_r}$$

$$Z = \int Dx_r \, Dx_a \, e^{i \int x_a(t) \frac{\delta S}{\delta x}\big|_r} = \int Dx_r \, \delta\left[\frac{\delta S}{\delta x_r}\right]$$

↑ at linear order in x_a get classical EOM

Now lets compute for the HO:

$$iS_1 = i\int dt \; \frac{1}{2}\dot{x}_1^2 - \frac{1}{2}\omega_0^2 x_1 + F_1 x_1$$

So

$$iS_1 - iS_2 = i\int dt \; \dot{x}_r \dot{x}_a - \omega_0^2 x_r x_a + F_r x_a + F_a x$$

Response Fcns:

$$iG_{ra}(t,\bar{t}) = \int Dx_r Dx_a \, e^{iS_1 - iS_2} \, x_r(t)\, x_a(\bar{t})$$

① $a \longrightarrow r$ $\quad iG_{ra} \equiv iG_R(t,\bar{t}) = \theta(t-\bar{t}) \langle [\hat{x}(t), \hat{x}(\bar{t})]$

Note:

$$-\left(\frac{d^2}{dt^2} + \omega_0^2\right) G_{ra}(t,\bar{t}) = \delta(t-\bar{t})$$

And Boundary condition $\langle x_r(t)\, x_a(\bar{t}) \rangle =$ vanishes at whenever $\bar{t} > t$

$$G_R = \frac{1}{\omega^2 - \omega_0^2 + i\varepsilon\omega}$$

② $G_A = G_{ar}(t,\bar{t})$ ← adwadced propagator

$$-\left(\frac{d^2}{dt^2} + \omega_0^2\right) G_A = \delta_{tt'}$$

③ Spectral Density

$$\rho(t,\bar{t}) = i(G_R - G_A) = \langle [\hat{x}(t), \hat{x}(\bar{t})] \rangle$$

$$\rho(\omega) = -2 \,\mathrm{Im}\, G_R(\omega)$$

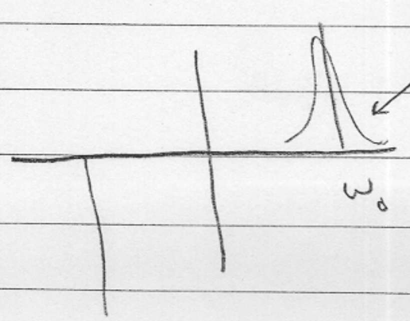
Note:

$$-\left(\frac{d^2}{dt^2} + \omega_0^2\right) \rho(t,t) = 0$$

Initial conditions $\rho(t,\bar{t})\Big|_{t=\bar{t}} = 0$

$$\partial_t \rho \Big|_{t=\bar{t}} = -i$$

Harmonic Oscillator



Fluctuations

$$G_{rr}(t,\bar{t}) = \langle x_r(t)\, x_r(\bar{t}) \rangle = \frac{1}{2} \langle \{ \hat{x}(t)\, \hat{x}(\bar{t}) \} \rangle$$

$$G_{rr}(t,\bar{t}) = \int \rho(x_1^0, x_2^0) \int_{x_1^0 x_2^0} Dx_r\, Dx_a\, e^{i\int dt\, \dot{x}_r \dot{x}_a - \omega_0^2 x_r x_a} \quad x_r(t)\, x_r(\bar{t})$$

Now to compute:

① $$W(x,p) = \int dx_a\, \rho\left(x_r + \frac{x_a}{2}, x_r - \frac{x_a}{2}\right) e^{ipx_a}$$

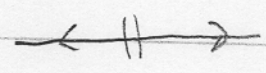
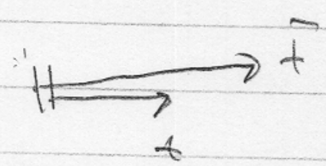
② $$x_r^{cl}(t) = m\left[X^0 \overleftrightarrow{\partial}_{t_0} G_R(t,t_0) - \partial_{t_0} x^0\, G_R(t,t_0)\right]$$
$$= m\left[x^0 \partial_{t_0} G_R(t,t_0)\right]$$

Then

$$G_{rr} = \int \frac{dx\, dp}{2\pi}\, W(x_r^0, p)\, x_r^{cl}(t)\, x_r^{cl}(\bar{t})$$

$$p = \partial_t x^0$$

So

G_{rr} =  or 

For a <u>thermal</u> state

$$\rho = e^{-\beta E_n} |n\rangle\langle n|$$

Find

$$G_{rr}(\omega) = \left(\frac{1}{2} + n(\omega)\right) \underbrace{\left[\frac{2\pi}{2E_p} \delta(\omega-\omega_0) - \frac{2\pi}{2E_p} \delta(\omega-\omega_0\right.}_{\rho(\omega)}$$

↑ Fluctuation Dissipation Theorem

$$G_{rr}(\omega) = \left(\frac{1}{2} + n(\omega)\right) \rho(\omega)$$ ← modes are thermally occupied

or

$$G^{>}(\omega) = G^{<}(\omega) e^{\beta\omega}$$

Why FDT is important:

Consider photon emission and absorption from sQG

$$2E_p \left[\partial_t + v_p \frac{\partial}{\partial x} \right] f_p = \underbrace{-f_p \, \Pi^{>}(P)}_{\text{Loss}} + \underbrace{\Pi^{<}(P)(1+f_p)}_{\text{gain}}$$

$$\Pi^{>}(P) = FT_{rans} \langle J(t) J(0) \rangle$$

$$\Pi^{<}(P) = FT_{rans} \langle J(0) J(t) \rangle$$

Using: $\rho(P) = \Pi^{>} - \Pi^{<}$

$$\Pi^{>}(P) = e^{p^0/T} \, \Pi^{<}$$

Find: Do it!

positive def

$$2E_p \left(\partial_t + v_p \frac{\partial}{\partial x} \right) f_p = -\left(f_p - \frac{1}{e^{p^0/T} - 1} \right) \rho(P)$$

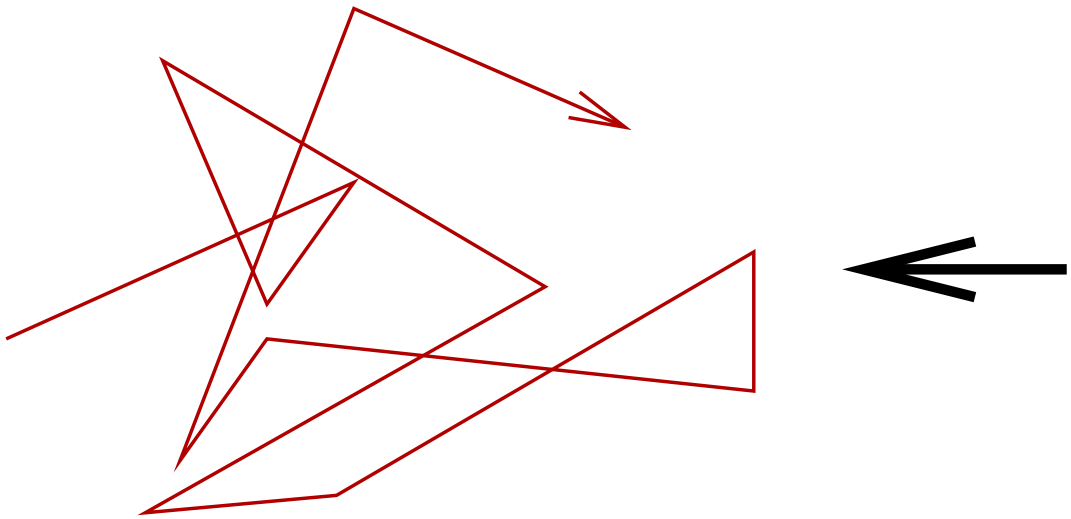
So f_p will evolve until it reaches equilibrium

Thermalization of fluctuations in strongly coupled plasmas

- Dam T. Son, DT; JHEP. arXiv:0901.2338
- Simon Caron-Huot, DT, Paul Chesler; PRD in press, arXiv:1102.1073

Heavy Quarks in equilibrium Quantum Field Theory

$$M\frac{d^2\mathbf{x}}{dt^2} = \underbrace{-\eta}_{\text{Drag}}\,\dot{\mathbf{x}} + \underbrace{\xi}_{\text{Noise}}$$



"Artist's" conception of Brownian Motion

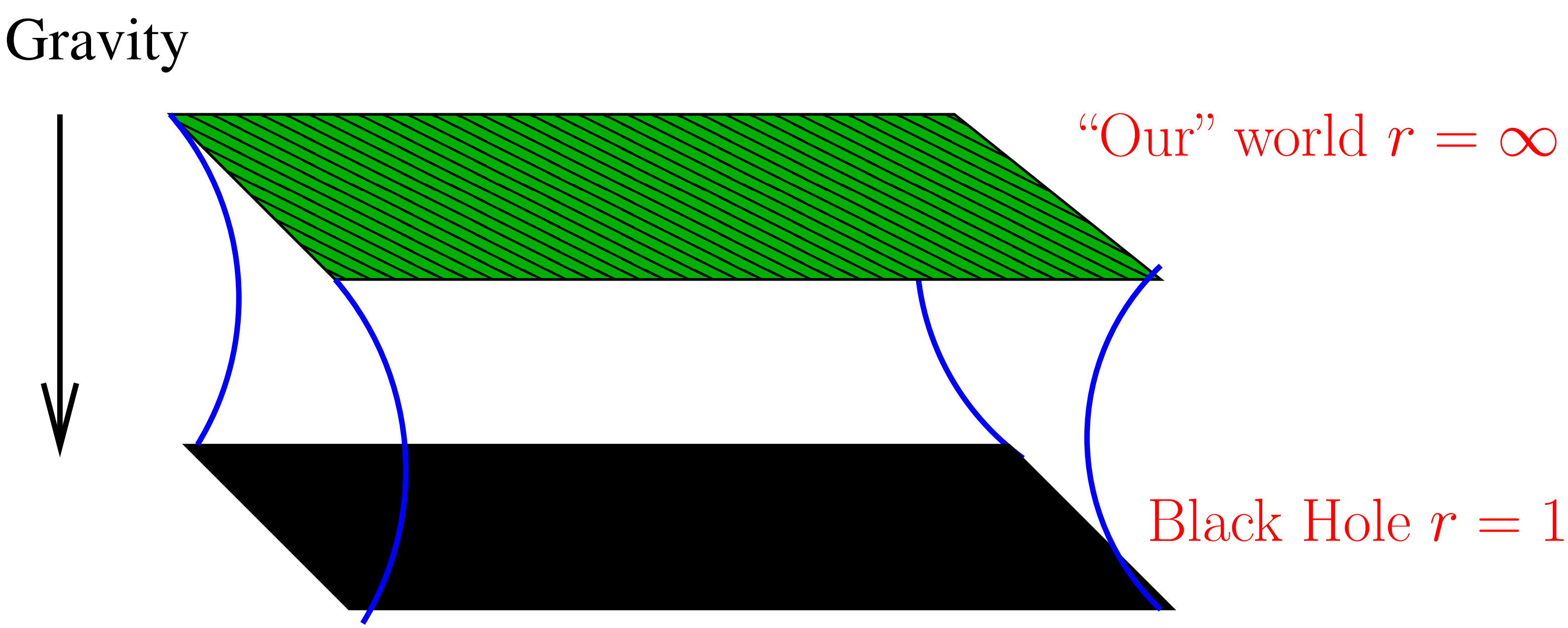
1. In equilibrium the drag and noise are balanced

$$\langle \xi(t)\xi(t')\rangle = 2T\eta\,\delta(t-t') \Leftarrow \text{Fluctuation Dissipation Theorem}$$

AdS/CFT

- Classical solutions in curved spacetime = CFT for nonzero temperature

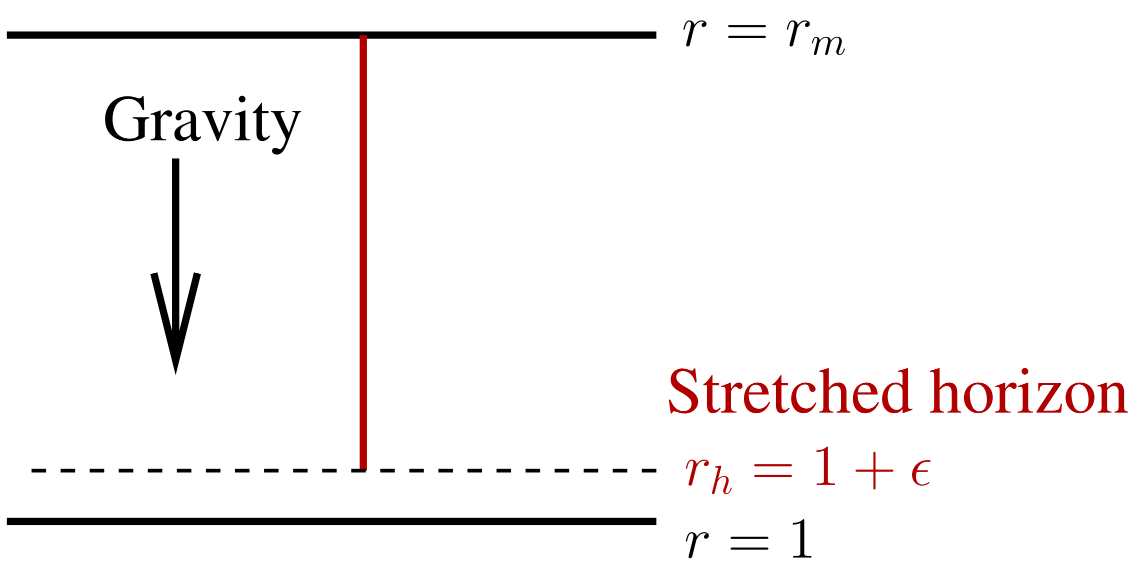
$$\mathrm{d}s^2 = (\pi T)^2 r^2 \left[-f(r)\mathrm{d}t^2 + \mathrm{d}x^2\right] + \frac{\mathrm{d}r^2}{r^2 f(r)} \qquad f(r) = 1 - \frac{1}{r^4}$$



How can a static metric be dual to equilibrium=constant fluctuations ?

Heavy Quarks in equilibrium AdS

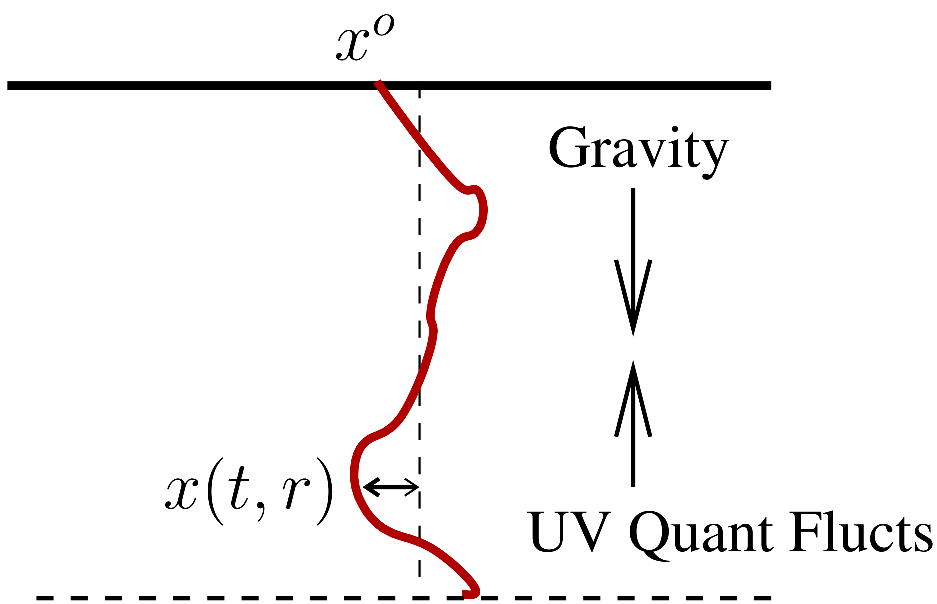
- Heavy quarks are classical strings in the 5d equilibrium AdS black hole geometry
- Solve classical string EOM and find:



Not the dual of an equilibrated quark!

Detailed Balance and Hawking Radiation:

$$M\frac{d^2x^o}{dt^2} = \underbrace{-\eta}_{\text{Drag}}\,\dot{x}^o + \underbrace{\xi}_{\text{Noise}}$$



Evolves to Classical

Prob Dist: (Son,DT;Iancu)

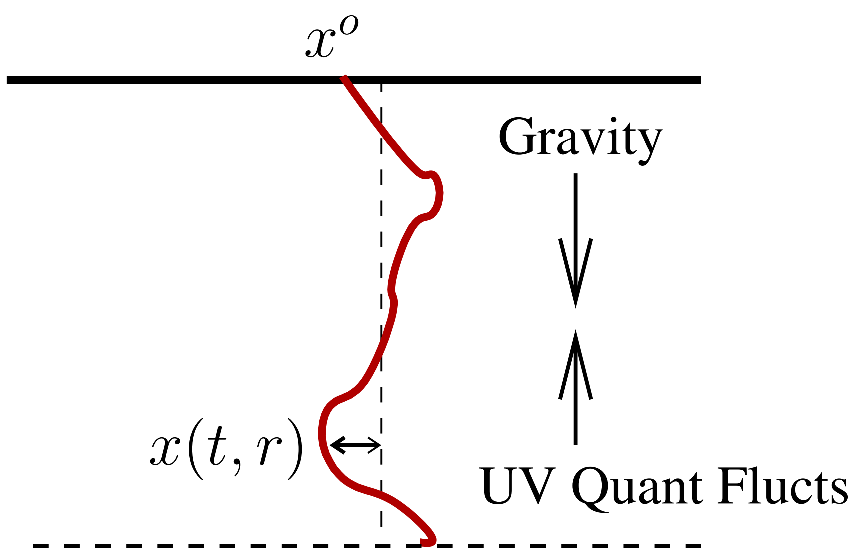
$$P[x,\pi_x] \propto e^{-\beta H[x,\pi_x]}$$

Goals:

1. Will show that Hawking Radiation is balanced by gravity

2. Generalize to non-equilibrium

Detailed Balance and Hawking Radiation (Technical Discussion)



1. Fluctuations:

$$G_{rr} \equiv \frac{1}{2} \langle \{ \hat{x}(t_1, r_1), \hat{x}(t_2, r_2) \} \rangle \ ,$$

2. Dissipation (Spectral Density)

$$\rho_{ra-ar} \equiv \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle \ .$$

- Equilibrium $\equiv$ Fluctuation Dissipation Theorem

$$G_{rr}(\omega, r_1, r_2) = \left(\frac{1}{2} + n_B(\omega) \right) \rho_{ra-ar}(\omega, r_1, r_2) \qquad n(\omega) \equiv \frac{1}{e^{\omega/T} - 1}$$

Formulas

- Action for string fluctuations, $h^{\mu\nu}$ = string metric

$$S_1 - S_2 = \frac{\sqrt{\lambda}}{2\pi} \int \mathrm{d}t\mathrm{d}r \, g_{xx} \left[-\sqrt{h} h^{\mu\nu} \partial_\mu x_r \partial_\nu x_a \right] \, ,$$

- $h^{\mu\nu}$ is the string metric

$$h_{\mu\nu} \mathrm{d}\sigma^\mu \mathrm{d}\sigma^\nu = -(\pi T)^2 r^2 f(r) dt^2 + \frac{dr^2}{f(r) r^2} \, ,$$

- Retarded Green Function

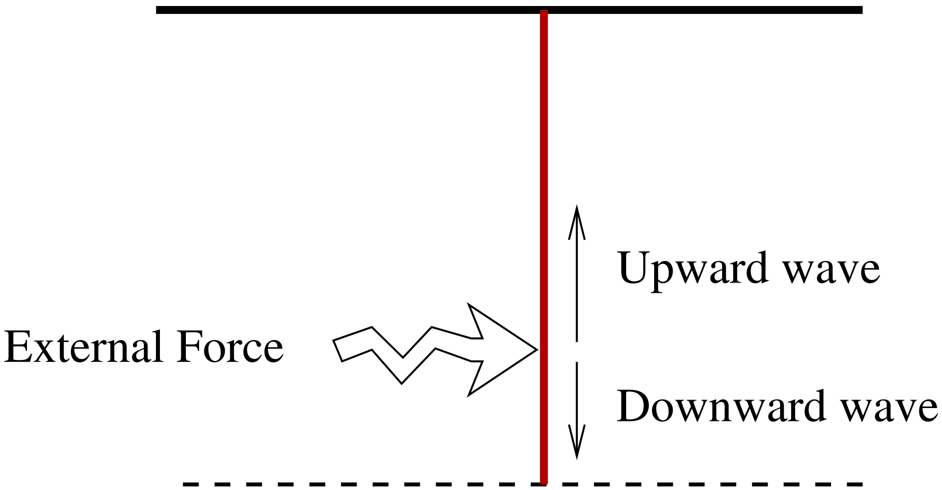
$$iG_{ra}(t_1 r_1 | t_2 r_2) \equiv \theta(t - t') \left\langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \right\rangle \, ,$$

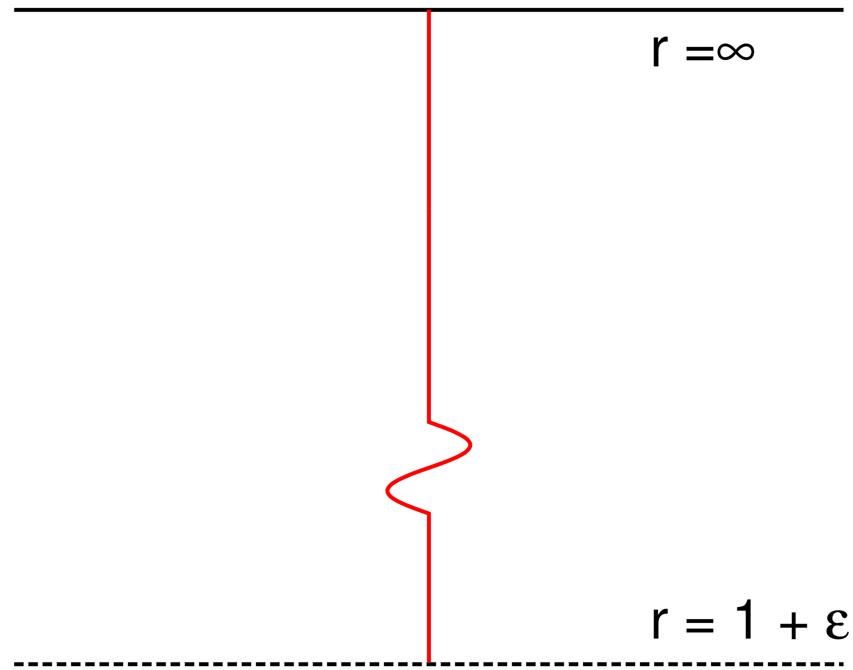
$G_{ra}(t_1 r_1 | t_2 r_2)$ is the classical response to a force at $t_2 r_2$

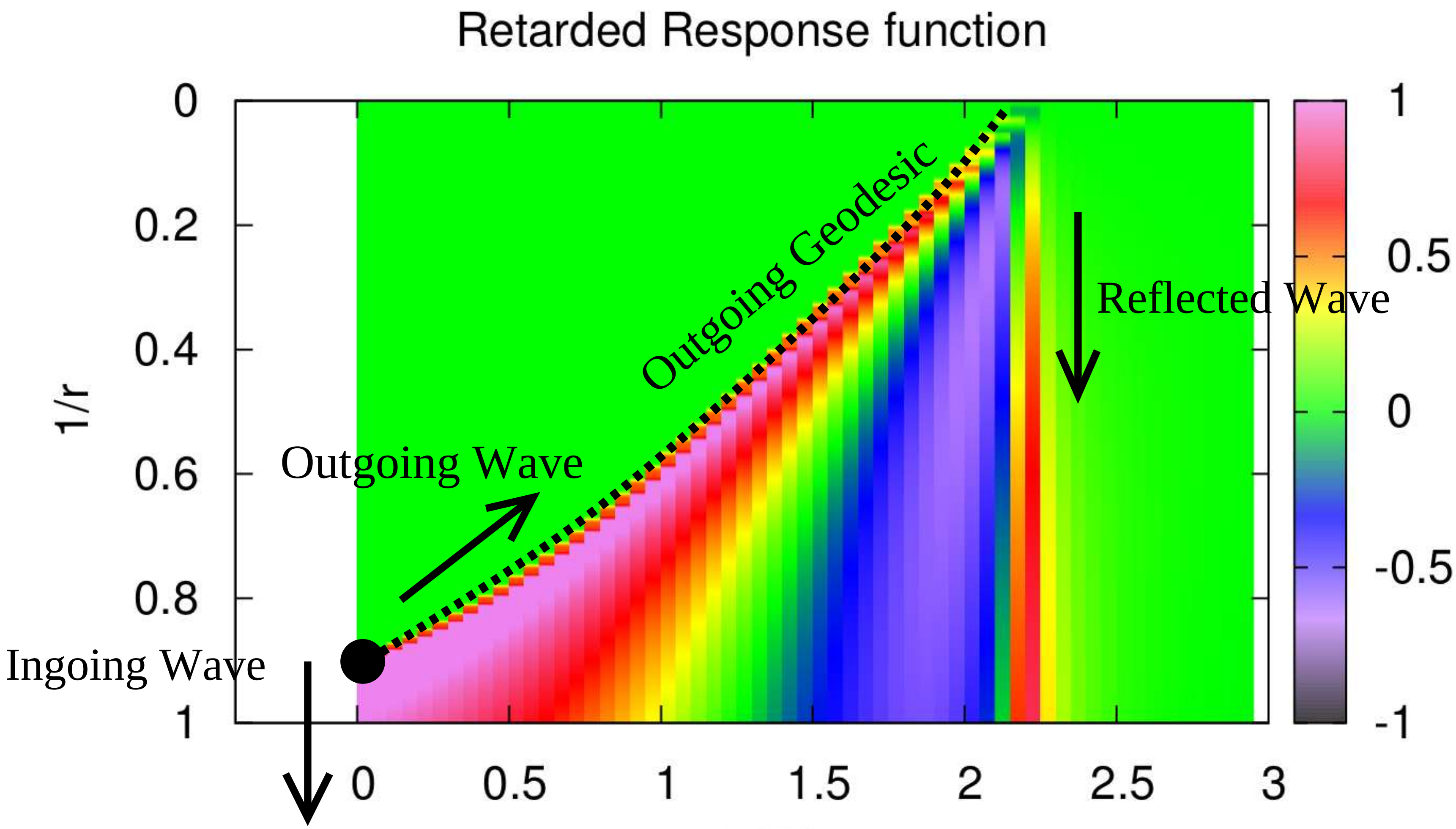
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu \, g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{ra}(t_1 r_1 | t_2 r_2) = \delta(t_1 - t_2) \delta(r_1 - r_2) \, ,$$

The classical Green Function or response to a force:

$$\frac{\sqrt{\lambda}}{2\pi}\left[\partial_\mu\, g_{xx}\sqrt{h}h^{\mu\nu}\partial_\nu\right] G = \mathcal{F}\,\delta(t_1 - t_2)\delta(r_1 - r_2)\,,$$





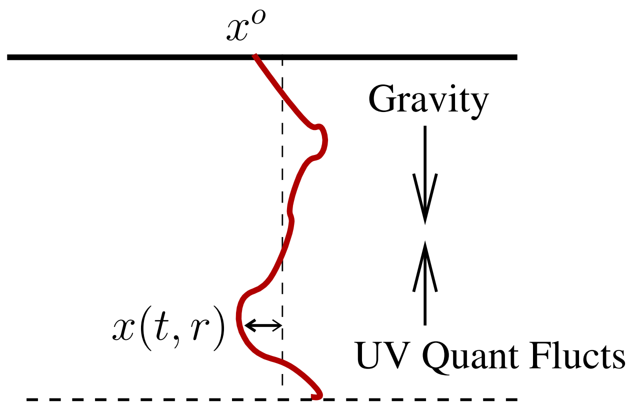


(Infalling Time)

$$v = t - \frac{1}{2\pi T}\left[\tan^{-1}(r) + \tanh^{-1}(r)\right]$$

v =Eddington time

Statistical Fluctuations



$$G_{rr} = \frac{1}{2} \langle \{ x(t_1, r_1), x(t_2, r_2) \} \rangle$$

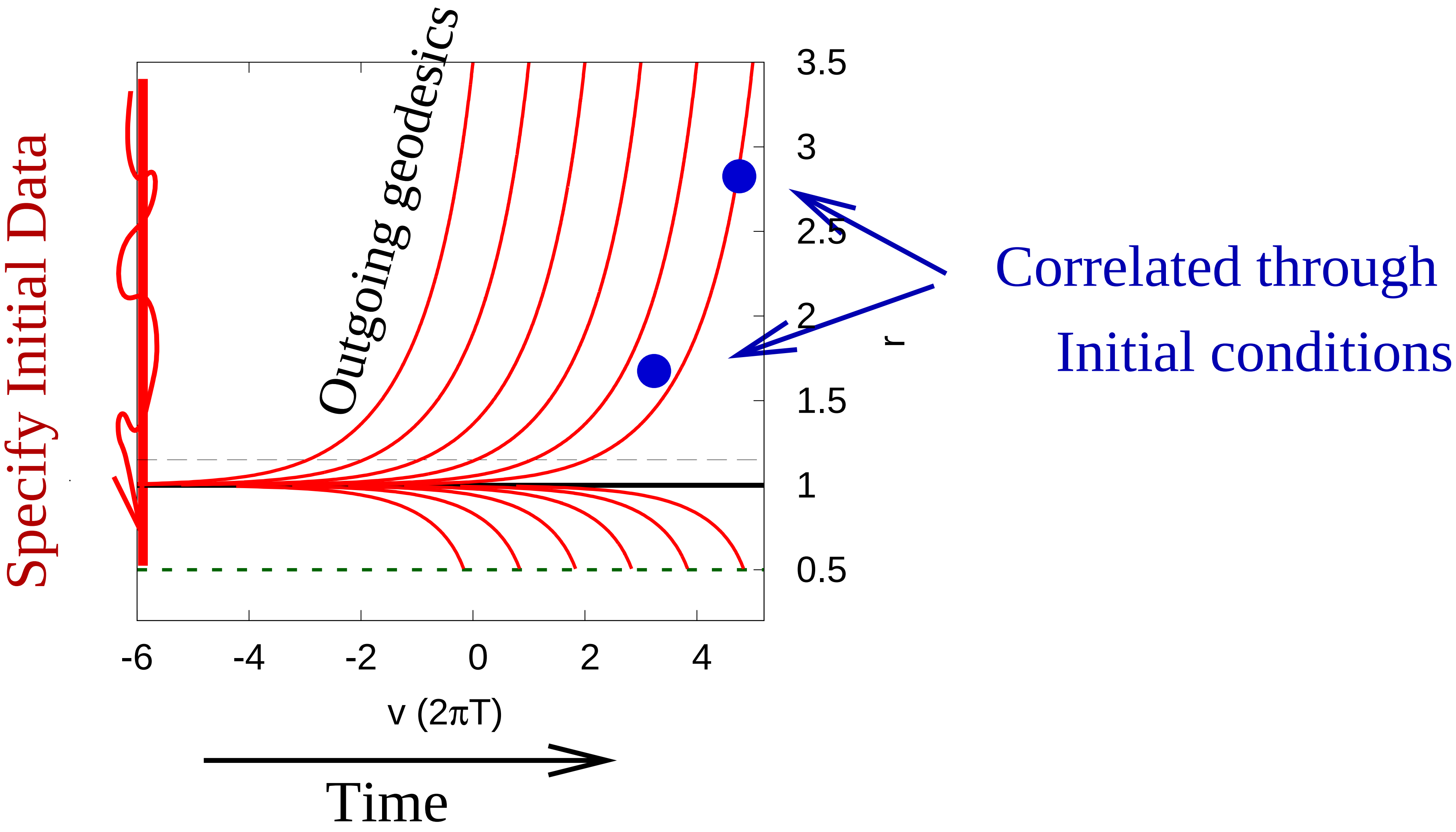
- The statistical correlator obeys the homogeneous EOM

$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu \, g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] G_{rr}(t_1 r_1 | t_2 r_2) = 0$$

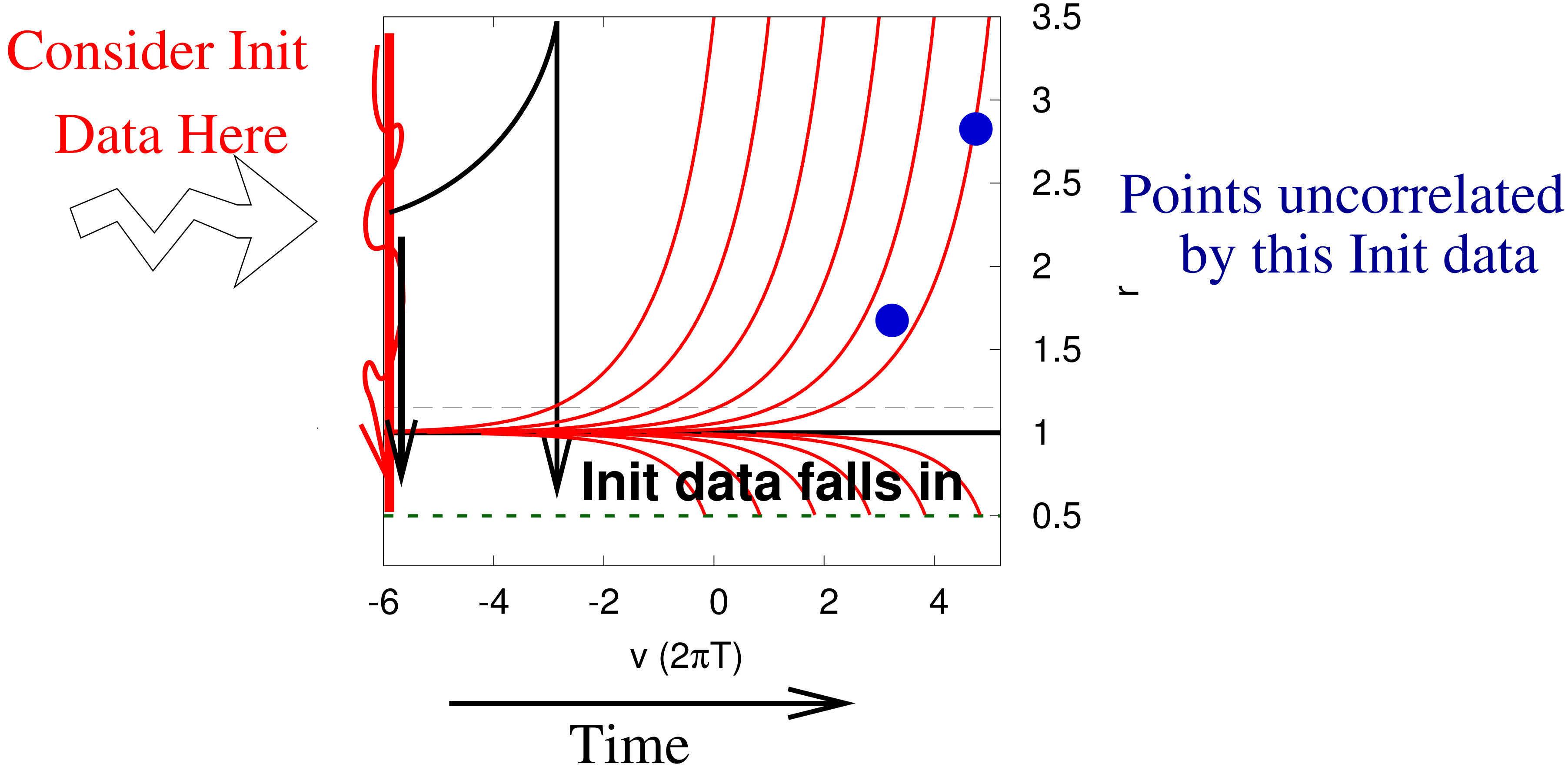
- So:

1. Specify the correlations (or density matrix) in the past
2. Final state fluctuations are correlated only through initial conditions

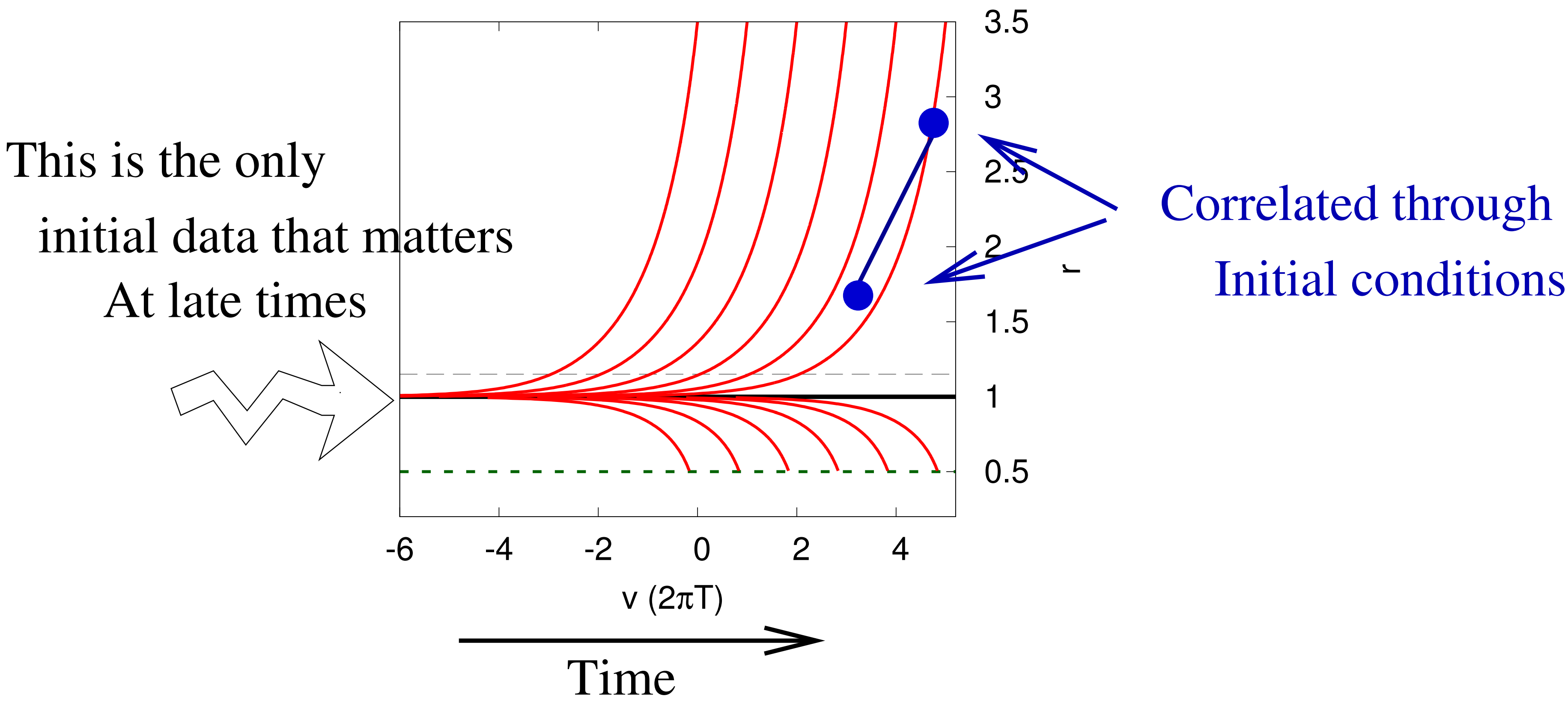
Correlations through Initial conditions



Correlations through Initial conditions



Correlations through Initial conditions



1. Final correlation come from correlated initial data very near horizon
    - Short Wavelength
2. Initial data is inflated by near horizon geometry

Initial Data from Quantum Fluctuations

1. Initial data is determined at short distance = Flat Space Physics

2. Scalar Field in 1+1D vacuum flat space

$$\frac{1}{2}\langle\{\phi(X_1),\phi(X_2)\}\rangle = -\frac{1}{4\pi K}\log|\mu\,\eta_{\mu\nu}\Delta X^{\mu}\Delta X^{\nu}|$$

K=norm of action

3. String flucts in near horizon geometry

$$S^{\rm near-horizon} = \eta\int {\rm d}t{\rm d}r\,\left[-\frac{1}{2}\sqrt{h}h^{\mu\nu}\partial_\mu x\partial_\nu x\right]$$

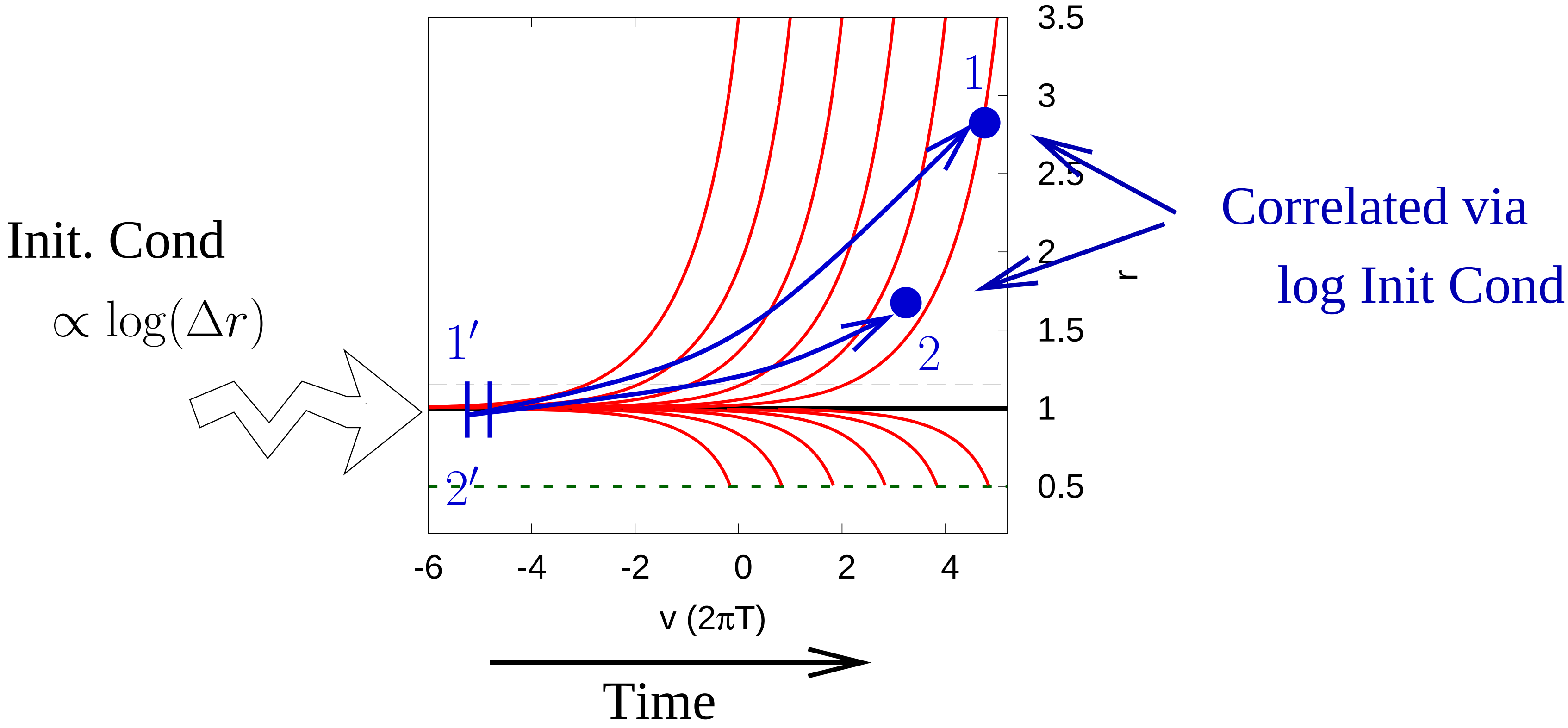
$\eta =$ Drag Coefficient

The near horizon initial condition is:

$$G_{rr}(v_1 r_1|v_2 r_2) \to -\frac{1}{4\pi\eta}\log\left|\mu\,\overbrace{2\Delta v\,\Delta r}^{\text{local }\Delta s^2}\right|$$

Summary: Specify IC and Solve Equations of Motion

$$\frac{\sqrt{\lambda}}{2\pi}\left[\partial_\mu\, g_{xx}\sqrt{h}h^{\mu\nu}\partial_\nu\right] G_{rr}(t_1r_1|t_2r_2) = 0$$

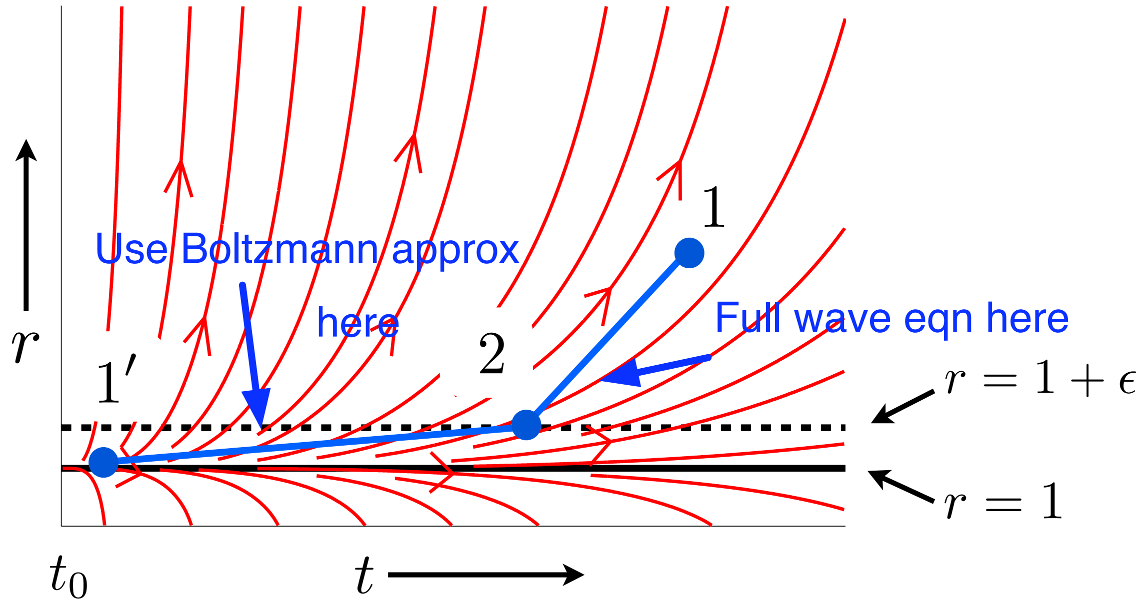


$$G_{rr}(1|2) = \left[\frac{\sqrt{\lambda}}{2\pi}\int dr_1'\; g_{xx}\sqrt{h}h^{tt}(r_1')\, G_{ra}(1|1')\overleftrightarrow{\partial}_{t_1'}\right]$$

Like Harmonic Osclator

$$\times\left[\frac{\sqrt{\lambda}}{2\pi}\int dr_2'\; g_{xx}\sqrt{h}h^{tt}(r_2')\, G_{ra}(2|2')\overleftrightarrow{\partial}_{t_2'}\right] G_{rr}(1'|2')\,,$$

From initial data to final correlations in two steps:



$$G_{ra}(1|1') = \int \mathrm{d}t_2\, G_{ra}(1|2) \left[\eta\sqrt{h}h^{rr}(r_2)\overleftrightarrow{\partial_{r_2}}\right]_{r_2=1+\epsilon} G_{ra}(2|1')\,,$$

(a) From horizon to stretched horizon – Waves are very short wavelength

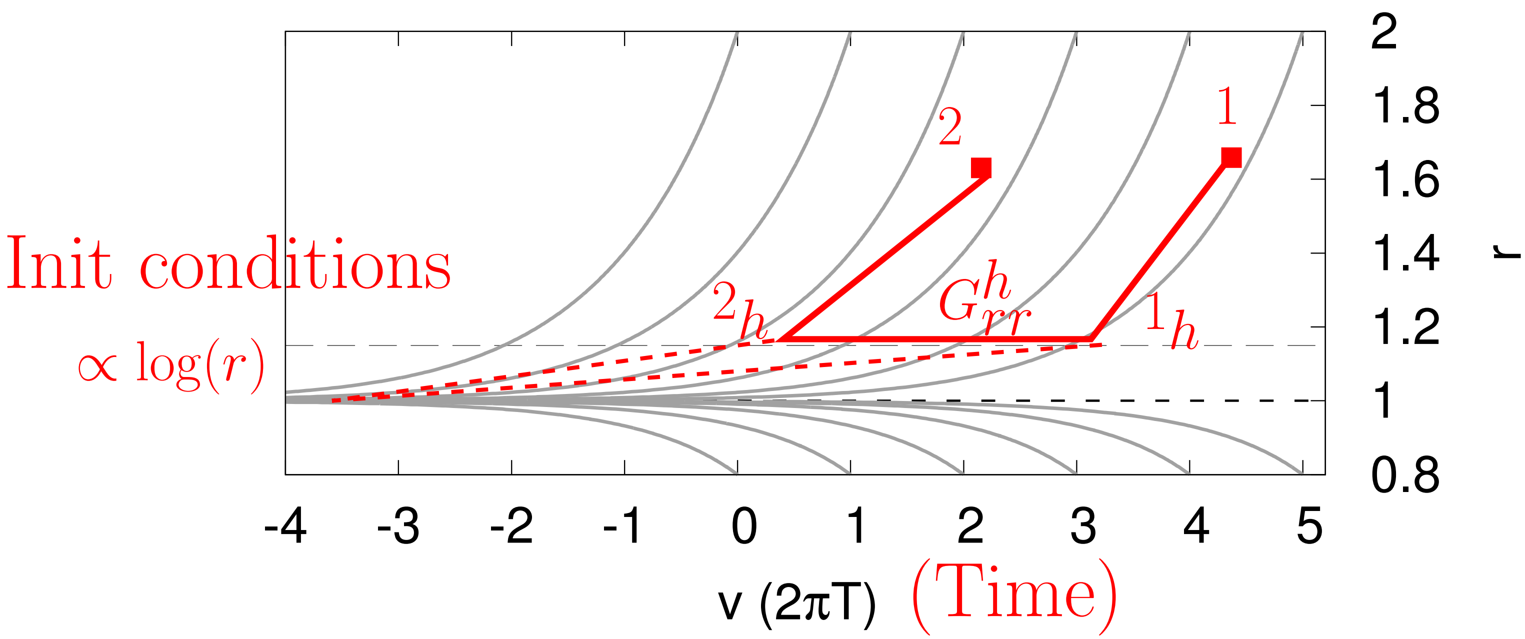
– Use collisionless Boltzmann approximation (geodesic/WKB/eikonal approx)

(b) The stretched horizon to boundary – Waves are longer wavelength

– Use full wave equation

Fluctuations from Equations of Motion

$$\underbrace{G_{rr}(1|2)}_{\text{bulk flucts}} = \int \mathrm{d}t_{1h}\mathrm{d}t_{2h} \underbrace{G_R(1|1_h)\, G_R(2|2_h)}_{\text{outgoing Green fcns}} \underbrace{G^h_{rr}(1_h|2_h)}_{\text{horizon flucts}},$$

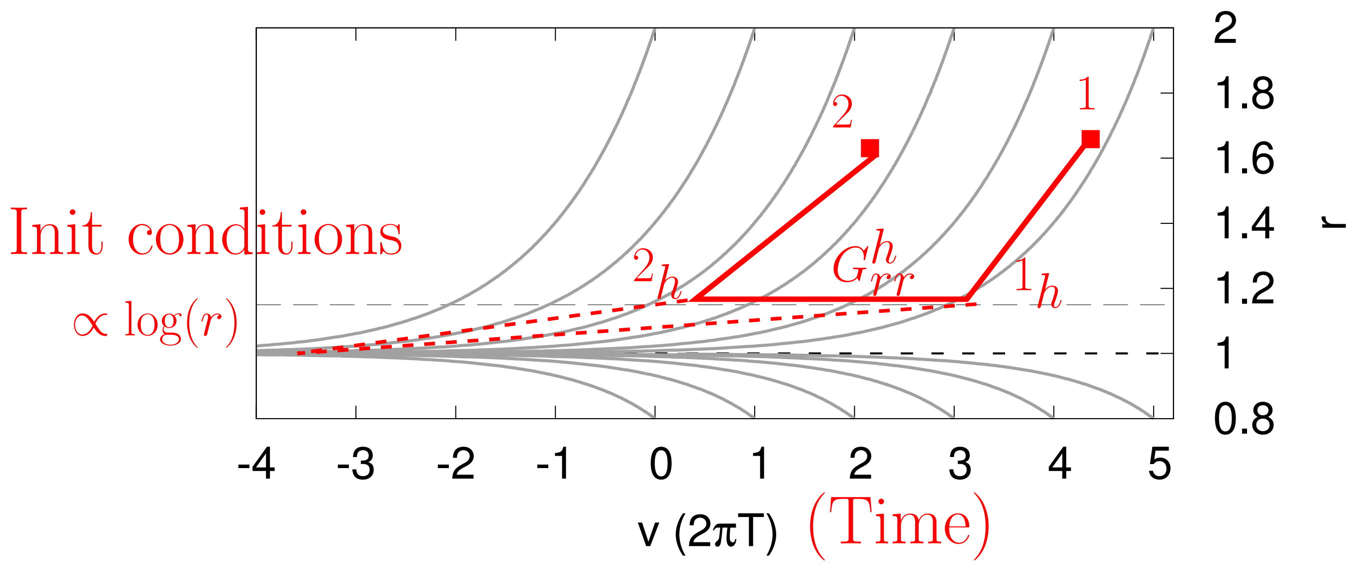


The fluctuations on the stretched horizon are from UV vacuum flucts in past

$$G^h_{rr}(t_1|t_2) = \text{Blow-up of initial data} \propto \log(r)$$

$$= -\frac{\eta}{\pi}\partial_{t_1}\partial_{t_2}\log\left|1 - e^{-2\pi T(t_1-t_2)}\right|.$$

The horizon fluctuations and the Lyapunov exponent



1. Thermal looking:

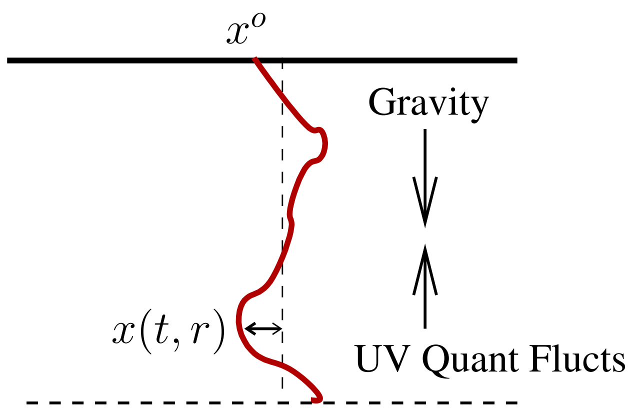
$$G^h_{rr}(\omega) = \text{Fourier-Trans of } -\frac{\eta}{\pi}\partial_{t_1}\partial_{t_2}\log|1 - e^{-2\pi T(t_1 - t_2)}|$$

$$= \left(\tfrac{1}{2} + n(\omega)\right) 2\omega\eta \qquad\qquad n(\omega) \equiv \frac{1}{e^{\omega/T} - 1}$$

2. Temperature $\propto$ inflation rate

$$2\pi T = \text{Lyapunov exponent of diverging geodesics}$$

Dissipation - Spectral Density



$$\rho_{ra-ar} = \langle [\hat{x}(t_1, r_1), \hat{x}(t_2, r_2)] \rangle$$

- The spectral density <u>also</u> obeys the EOM

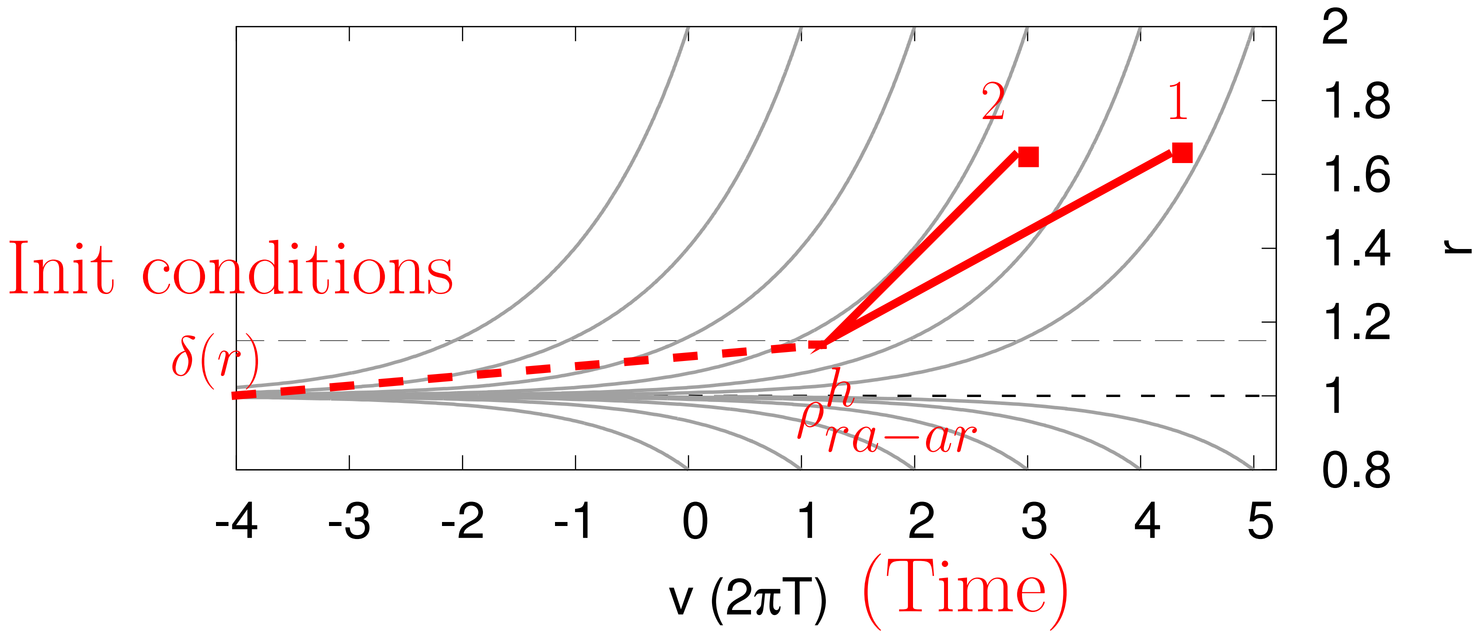
$$\frac{\sqrt{\lambda}}{2\pi} \left[\partial_\mu \, g_{xx} \sqrt{h} h^{\mu\nu} \partial_\nu \right] \rho_{ra-ar}(t_1 r_1 | t_2 r_2) = 0$$

- <u>But</u> initial conditions are given by the canonical commutation relations

$$\eta \sqrt{h} h^{tt}(r_1) \lim_{t_2 \to t_1} \partial_{t_1} \rho_{ra-ar}(t_1 r_1 | t_2 r_2) = i\delta(r_1 - r_2) \, .$$

Spectral Density

$$\underbrace{\rho_{ra-ar}(1|2)}_{\text{bulk spectral fcn}} = \int \mathrm{d}t_{1h}\mathrm{d}t_{2h}\, \underbrace{G_R(1|1_h)\; G_R(2|2_h)}_{\text{outgoing Green fcns}}\; \underbrace{\rho^h_{ra-ar}(1_h|2_h)}_{\text{horizon spectral fcn}} \;,$$



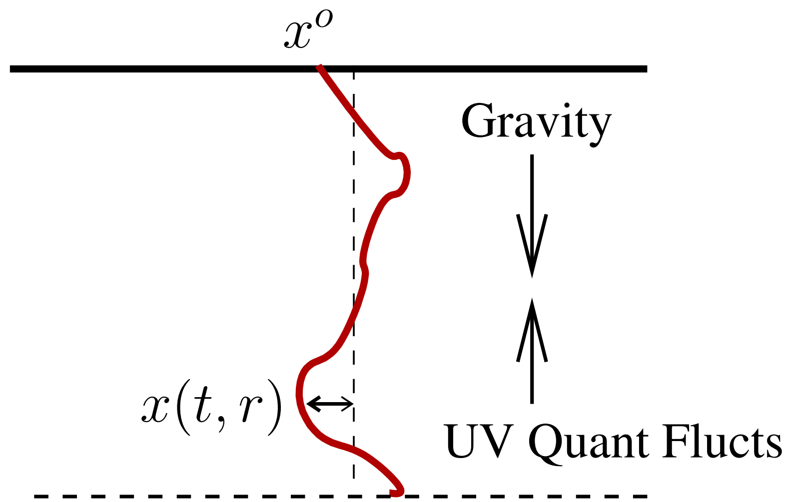
Where the horizon spectral density

$$\rho^h_{ra-ar}(t_1, t_2) = \text{local due to canonical commutation relations}$$

$$= 2\eta\left[-i\delta'(t_1 - t_2)\right] \qquad (2\omega\eta \text{ in Fourier space})$$

Detailed Balance

$$G_{rr}(\omega, r_1, r_2) = \left(\tfrac{1}{2} + n(\omega)\right) \rho(\omega, r_1, r_2)$$



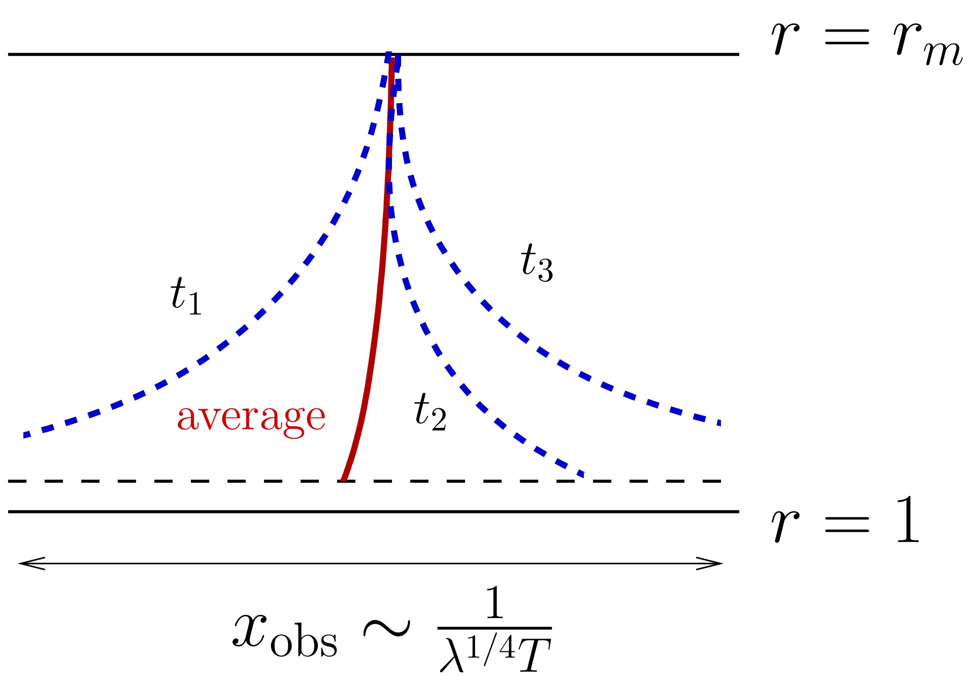
1. Fluctuations (Anti-commutator)

$$\underbrace{G_{rr}(\omega, r_1, r_2)}_{\text{bulk flucts}} = \underbrace{G_R(\omega, r_1|r_h)\; G_R(\omega, r_2|r_h)}_{\text{outgoing Green fcns}}\; \underbrace{\left(\tfrac{1}{2} + n(\omega)\right) 2\omega\eta}_{\text{Horizon-flucts}}$$

2. Dissipation: (Commutator)

$$\underbrace{\rho_{ra-ar}(\omega, r_1, r_2)}_{\text{bulk spec dense}} = \underbrace{G_R(\omega, r_1|r_h)\; G_R(\omega, r_2|r_h)}_{\text{outgoing Green fcns}}\; \underbrace{2\omega\eta}_{\text{Horizon spec dense}}$$

Fluctuation dissipation and stochastic dynamics



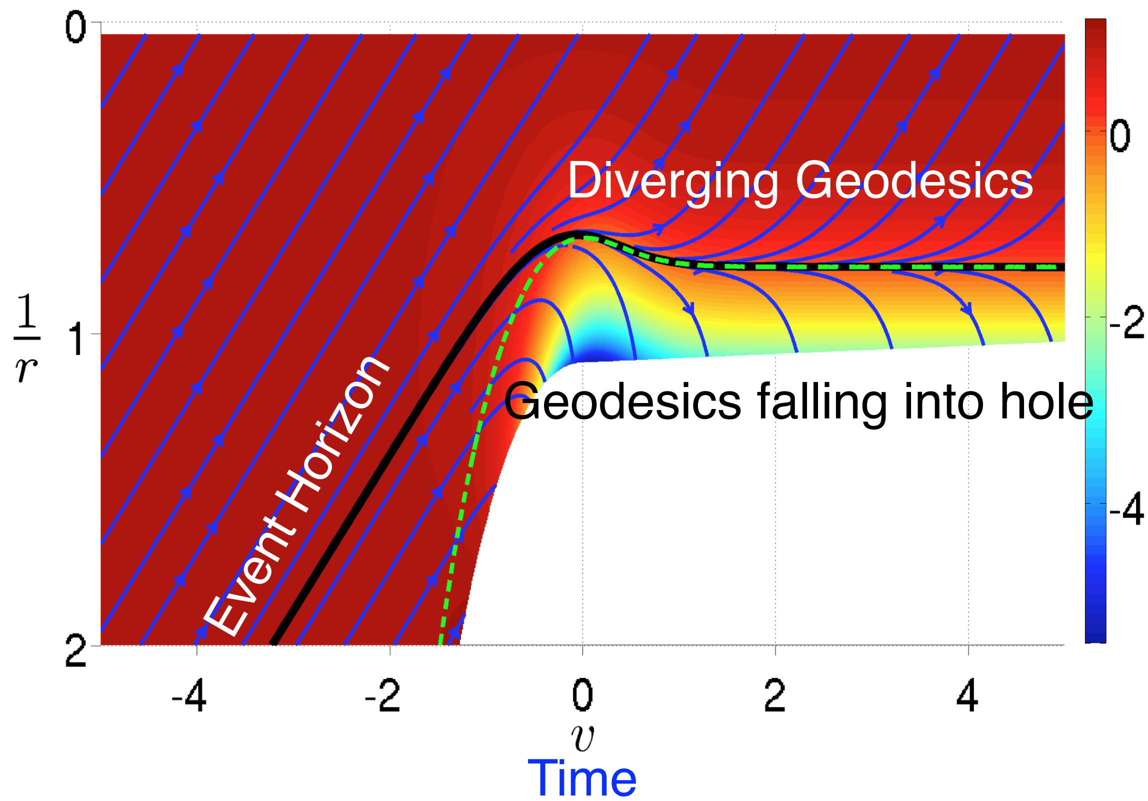
1. Every step t_1, t_2, t_3 fluctuates to a new trailing string – $\longrightarrow$ random force

2. The average of the trailing strings gives the drag – average string $\longrightarrow$ drag

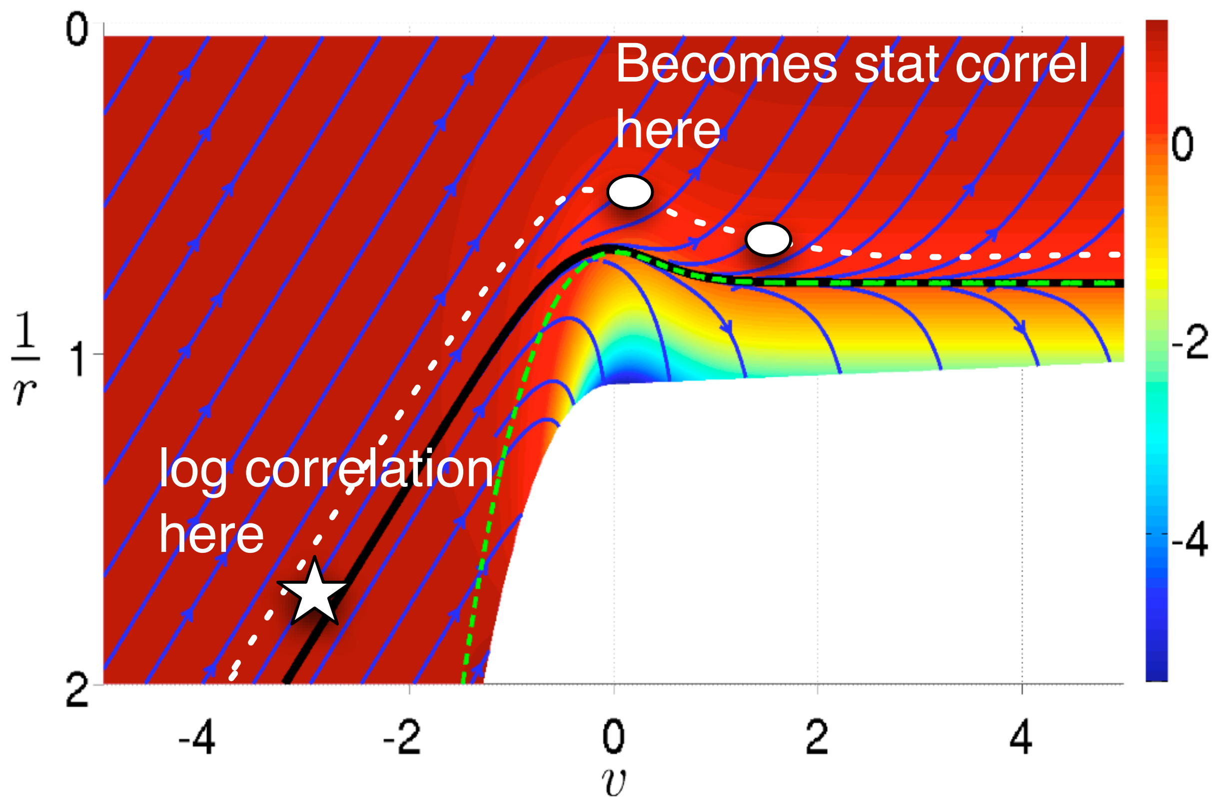
Non-equilibrium

1. Chesler&Yaffe create QGP by turning a gravitational pulse in vacuum

2. Corresponds to non-equilbrium geometry with BH formation in AdS_5



Fluctuations in non-equilibrium



- Surface Properties – on event horizon

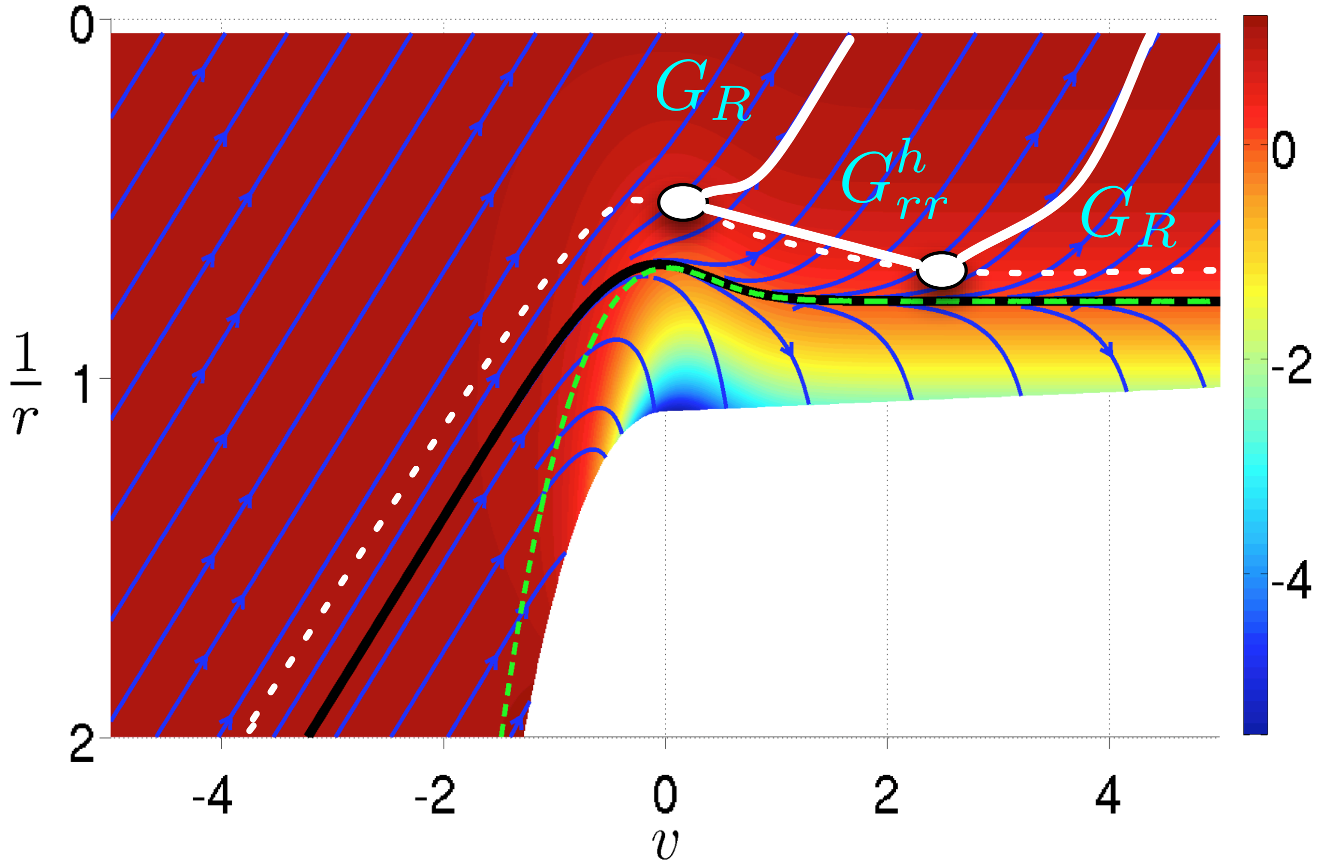
$$\underbrace{2\pi T_{\text{eff}}(v)}_{\text{Lyapunov exponent}} \equiv \overbrace{\frac{1}{2}\frac{\partial A(r,v)}{\partial r}}^{\text{Metric-coeff}}\Bigg|_{r=r_h(v)} \propto \text{extrinsic curvature}$$

Result:

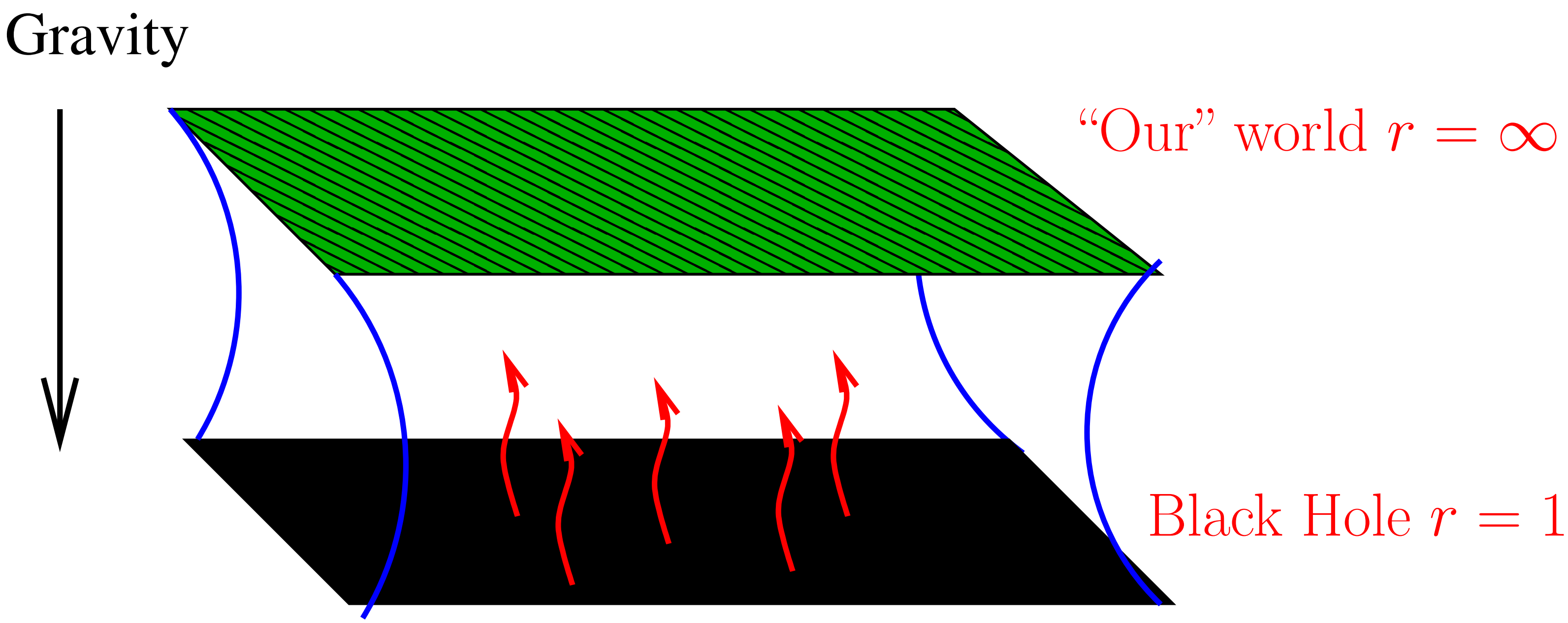
- General form of near horizon fluctuations in non-equilibrium

$$G^h_{rr}(v_1|v_2) = -\frac{\sqrt{\eta(v_1)\eta(v_2)}}{\pi}\partial_{v_1}\partial_{v_2}\log\left|1 - e^{-\int_{v_1}^{v_2} 2\pi T_{\rm eff}(v')dv'}\right|.$$

- Can map the near horizon fluctuations up to boundary (numerics in progress)



Not conclusions, but picture:



Gravity pulls down, but quantum fields fluctuate, reaching equilibrium