Ideal Hydrodynamics: Tensor Analysis

- Given a $T^{\mu\nu}$, there is a frame where $T^{0i} = 0$.

  Denote this Local Rest Frame (LRF) by $u^\mu = (\gamma, \gamma \mathbf{v})$ and $u^2 = -1$. $u^\mu = (1, \mathbf{0})$. $g^{\mu\nu} = (-,+,+,+)$.

- In the LRF the stress energy tensor in equilibrium is

$$T^{\mu\nu} = \begin{array}{c c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \end{array} \quad \text{and} \quad N^\mu = (n, \mathbf{0})$$

- $T^{\mu\nu}$ must be a function of $u^\mu u^\nu$ and $g^{\mu\nu}$.

  Can also use the projector $\Delta^{\mu\nu}$ and $u^\mu u^\nu$:

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu = \text{diag}(0, 1, 1, 1)$$

  Verify: $\Delta^{\mu\nu} u_\nu = 0$ and $\Delta \cdot \Delta = \Delta$

- The general vector and tensor having this form:

$$\begin{aligned} T^{\mu\nu} &= e u^\mu u^\nu + p\, \Delta^{\mu\nu} \\ N^\mu &= n\, u^\mu \end{aligned}$$

Ideal Hydrodynamics: Physics

- Equations of motion:

$$\begin{aligned} \partial_\mu T^{\mu\nu} = \partial_\mu(e\, u^\mu u^\nu + p\,\Delta^{\mu\nu}) &= 0 \\ \partial_\mu N^\mu = \partial_\mu(n u^\mu) &= 0 \end{aligned}$$

- Equation of State (EoS): $p(e,n)$

- Five Equations of Motion (T^{00}, T^{0i}, N^0) and Five unknowns $e, \mathbf{v}, n$.

To interpret these EOM let us write them in the LRF:

$$\begin{aligned} \partial_t T^{00} \rightarrow \partial_t e &= -(e+p)\,\partial_i v^i \\ \partial_t T^{0i} \rightarrow \partial_t v_i &= -\frac{1}{e+p}\,\partial_i p \\ \partial_t N^0 \rightarrow \partial_t n &= -n\,\partial_i v^i \end{aligned}$$

- The acceleration is proportional to pressure gradients. The “mass”is replaced by enthalpy in a relativistic theory .

- Equations of motion depend only on the gradients of pressure and enthalpy $\equiv e + p$. Independent of Bag constant.

Adiabatic Expansion:

$$\frac{dV}{V} = dt \times \partial_i v^i$$

$$\begin{aligned} \partial_t e &= -(e+p)\,\partial_i v^i \\ \partial_t n &= -n\,\partial_i v^i \end{aligned}$$

$$\begin{aligned} de &= -(e+p)\frac{dV}{V} & \Longrightarrow \quad d(eV) &= -pdV \\ dn &= -n\frac{dV}{V} & \Longrightarrow \quad d(nV) &= 0 \end{aligned}$$

Compare : $d(eV) = Td(sV) - pdV - \mu d(nV)$ and find

$$d(sV) = 0 \ \text{ or } \ \partial_\mu(su^\mu) = 0 \ \text{ or } \ u^\mu \partial_\mu (n/s) = 0$$

- The system evolves on the Adiabatic path, $n/s =$ const.
- Important thermodynamic relations exist for this trajectory.

$$\begin{aligned} \left(\frac{dp}{de}\right)_{n/s} &\equiv c_s^2 \\ \left(\frac{ds}{de}\right)_{n/s} &= \frac{s}{e+p} \end{aligned}$$
(The second law, Verify)

- Given c_s^2, these equations can be integrated to determine the entropy and all other thermodynamic quantities.

Ideal Hydrodynamics: Some Notation

$$D = u^{\alpha}\partial_{\alpha} = \text{The time derivative in the rest frame}$$

$$\nabla^{\mu} = \Delta^{\mu\nu}\partial_{\nu} = \text{The spatial derivative in the rest frame}$$

Then the ideal equations of motion ($\partial_{\mu}T^{\mu\nu} = 0$) be written

$$\begin{aligned} De &= -(e+p)\nabla_{\mu}u^{\mu} \\ Du^{\mu} &= -\frac{1}{e+p}\nabla^{\mu}p \\ Dn &= -n\nabla_{\mu}u^{\mu} \end{aligned}$$

Verify: these equations from $\partial_{\mu}T^{\mu\nu} = 0$:

Use: $\partial_{\mu} = u_{\mu}D + \nabla_{\mu}$, $u^2 = -1$, $\Delta^{\mu\nu}u_{\nu} = 0$,
$u_{\mu}Du^{\mu} = u_{\mu}\nabla^{\alpha}u^{\mu} = 0$.

This notation facilitates easy going back and forth between LRF and covariant notation:

$$D \to \partial_t \quad \text{and} \quad \nabla_{\mu} \to \partial_i$$

Verify: Entropy conservation conservation D(s/n) = 0 .

Example: Entropy Conservation

$$
\begin{aligned}
D(s/n) &= \frac{1}{n}\left(\frac{\partial s}{\partial e}\right)_n De + \left\{\frac{1}{n}\left(\frac{\partial s}{\partial n}\right)_e - s/n\right\}\frac{Dn}{n} \\
&= -\left[\frac{e+p}{nT} - \frac{\mu}{nT} - s/n\right]\nabla_\mu u^\mu \\
&= 0 \qquad \left(\text{with } \; e + p - sT = \mu n\right)
\end{aligned}
$$

Viscous Hydrodynamics: Tensor Analysis

Consider the EOM close to the rest frame, $\mathbf{u^i} \ll \mathbf{1}$

- $T^{00} = e, T^{0i} = M^i = (e+p)u^i\ , N^0 = n$

$$\begin{aligned} \partial_t e + \partial_i M^i &= 0 \\ \partial_t M + \partial_i \tau^{ij} &= 0 \\ \partial_t n + \partial_i j_n &= 0 \end{aligned}$$

Determine the form of τ^{ij} and j^i by expanding around equilibrium to first order in gradients: $\partial_i M_j\ ,\ \partial_i e\ ,\ \partial^i n$

Vector	$\partial^i n\ ,\ \partial^i e$
Tensor	$\partial^i M^j + \partial^j M^i - \frac{2}{3}\delta^{ij}\partial_l M^l\ ,\ \delta^{ij}\partial_l M^l$
Pseudo Vector	$\epsilon^{ijk}\partial_j M_k$ Needed if system is rotating

$$\begin{aligned} j_e^i &= M^i \text{ (By symmetry } T^{0i} = T^{0i}\text{)} \\ \tau^{ij} &= p\,\delta^{i,j} + a_0\,\delta^{ij}\partial_i M^i \\ &\quad + a_1\,(\partial^i M^j + \partial^j M^i - \frac{2}{3}\delta^{ij}\partial_l M^l) \\ j^i &= nv^i + b_0\partial^i e + b_1\partial^i n \end{aligned}$$

(Aside: Statistical Mechanics with Motion)

$$\text{(Energy)} \qquad dE = TdS - pdV - v_i dM^i$$

$$\text{(Partition Function)} \qquad Z \propto \sum_n e^{-\frac{E_n - v_i M_n^i}{T}}$$

It is better to expand in gradients of thermodynamic conjugates to exploit Onsager's reciprocity relations:

$$\begin{aligned} ds &= \left(\frac{\partial s}{de}\right)_{n,M^i} de + \left(\frac{\partial s}{dn}\right)_{e,M^i} dn + \left(\frac{\partial s}{\partial M^i}\right)_{e,n} dM^i \\ &= \frac{1}{T}\, de - \frac{\mu}{T}\, dn - \frac{v_i}{T}\, dM^i \end{aligned}$$

We use the gradients $\partial^i \frac{1}{T}$, $\partial^i \left(\frac{\mu}{T}\right)$, and $\partial^i u^j$.

(Note $\partial^i \left(\frac{u^j}{T}\right) \approx \frac{1}{T}\partial^i u^j$ since $u^i \ll 1$)

$$\begin{aligned} j_e^i &= M^i + [\lambda_{ee} = 0] + \underbrace{[\lambda_{en} = 0]} \\ \tau^{ij} &= p\delta^{i,j} + \lambda_{MM}^L\, \delta_{ij}\partial_l \left(\frac{u^l}{T}\right) \\ &\quad + \lambda_{MM}^T \left(\partial^i u^j + \partial^j u^i - \frac{2}{3}\delta^{ij}\partial_l u^l\right) \\ j_n^i &= nu^i + \lambda_{nn}\, \partial^i \left(\frac{\mu}{T}\right) + \underbrace{\lambda_{ne}\, \partial^i \left(\frac{1}{T}\right)} \end{aligned}$$

Onsager Reciprocity: $\lambda_{ne} = \lambda_{en} = 0$.

Ingredients in proof of Onsager's Reciprocity:

- Symmetry (or Anti-Symmetry) under Time reversal. Needs modification when a Magnetic field is present.
- Entropy is a maximized in equilibrium and hence is a quadratic function of e, n, m^i.
- Symmetry + Onsager are needed to fix the form of the stresses.

Boosting the results:

1. First write

$$
\begin{array}{rcl|rcl}
T^{\mu\nu} & = & T_0^{\mu\nu} + \tau^{\mu\nu} & \tau^{\mu\nu} u_\nu & = & 0 \quad (T^{\mu\nu} u_\nu = e u^\mu) \\
N^\mu & = & N_0^\mu + q^\mu & q^\nu u_\nu & = & 0 \quad (N^\nu u_\nu) = n)
\end{array}
$$

2. The Spatial gradients are:

$$\nabla^\mu u^\nu \ , \ \nabla^\mu T \ , \ \nabla^\mu \left(\frac{\mu}{T}\right)$$

3. Expand $\tau^{\mu\nu}$ and q^μ

$$
\begin{aligned}
\tau^{\mu\nu} &= -\eta \left\langle \nabla^\mu u^\nu \right\rangle - \sigma \Delta^{\mu\nu} \nabla_\alpha u^\alpha \\
q^\mu &= -\kappa \left(\frac{nT}{e+p}\right)^2 \nabla^\mu \left(\frac{\mu}{T}\right) + \underbrace{[\lambda_{ne} = 0]} \partial^\mu \left(\frac{1}{T}\right)
\end{aligned}
$$

Definition: $\langle \nabla^\mu u^\nu \rangle \equiv \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla_\alpha u^\alpha$

Entropy production: (Ignore baryon number for simplicity)

- Definition: $T^{\mu\nu}u_\nu = eu^\mu$
- Differentiate: $\partial_\mu T^{\mu\nu}u_\nu$. Find

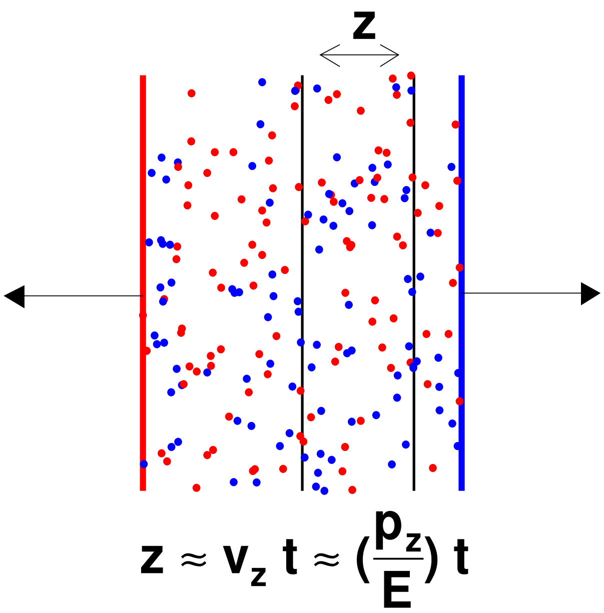
$$\underbrace{De = -(e+p)\nabla_\mu u^\mu}_{T\partial_\mu(su^\mu)} - \tau^{\mu\nu}\partial_\mu u_\nu$$

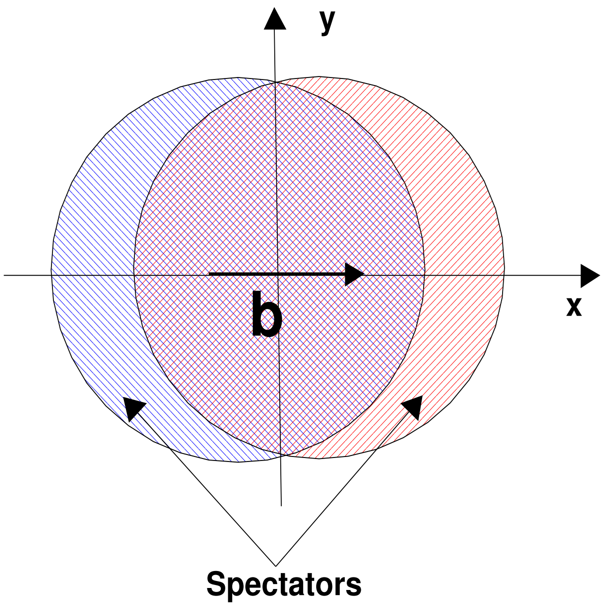
- Thermodynamic Relations: $\frac{ds}{de} = \frac{1}{T}$ and $sT = e + p$
- Put It all together: Find

$$\partial_\mu(su^\mu) = \frac{\eta}{2T}\langle\nabla^\mu u^\nu\rangle\langle\nabla_\mu u_\nu\rangle + \frac{\sigma}{T}(\nabla_\alpha u^\alpha)^2$$

- Entropy production = $\frac{1}{LT}$ $\times$ viscous coefficients.
- Require: $\eta > 0, \sigma > 0$ and $\kappa > 0$

A View of Heavy Ion Collision: The Bjorken expansion





Define the space time rapidity and proper time:

$\eta_s = \frac{1}{2}\log\frac{1+z/t}{1-z/t}$ and $\tau = \sqrt{t^2 - z^2}$

$$\eta_s \equiv \frac{1}{2}\log\frac{1+z/t}{1-z/t} \approx \frac{1}{2}\log\frac{1+p_z/E}{1-p_z/E} \equiv y$$

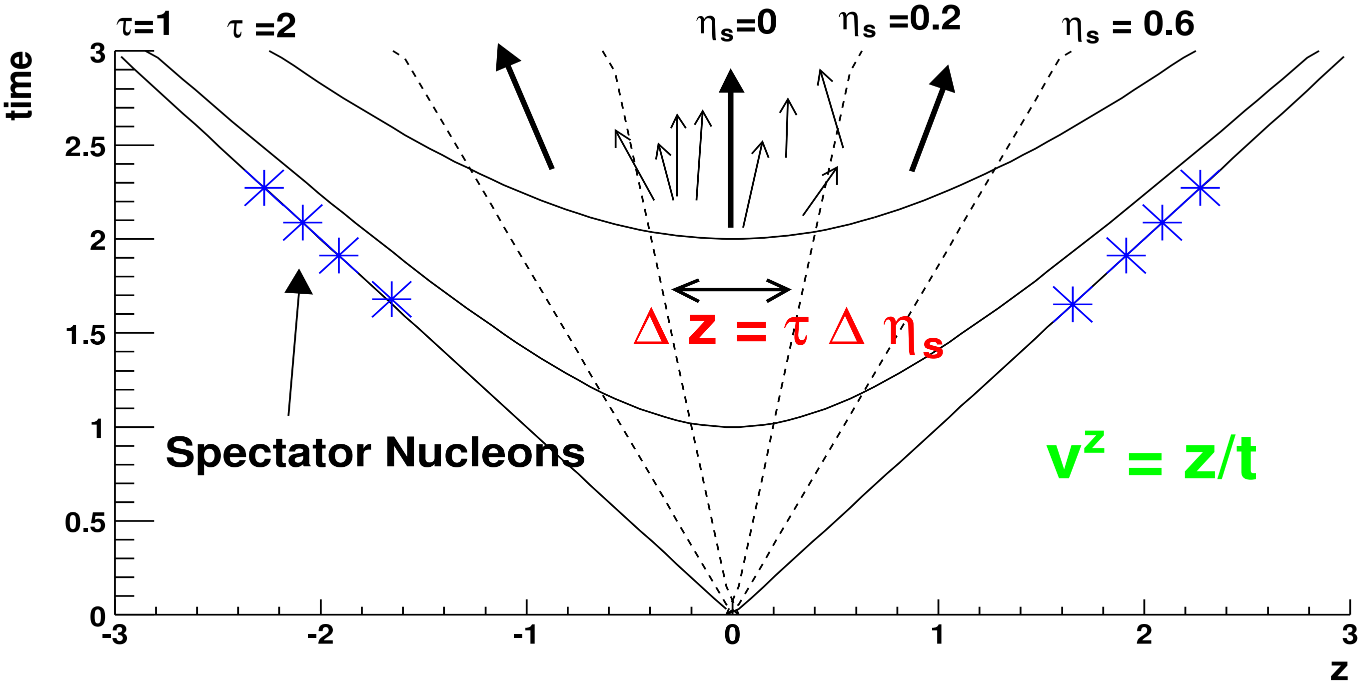
Up to smearing, all rapidities are the same in high energy collision

For hydrodynamics this means

- This means for the fluid $\langle v_z \rangle = \frac{u^z}{u^0} = z/t = \tanh(\eta_s)$. Or

$$u^z(\tau,\eta_s) = \sinh(\eta_s) \;\text{ and }\; u^0(\tau,\eta_s) = \cosh(\eta_s)$$

- Look for solutions of $\partial_\mu T^{\mu\nu} = 0$ of this form.



Plug in $u^\mu = (u^0, u^z) = (\cosh(\eta_s), \sinh(\eta_s))$

$$\begin{aligned} De &= -(e+p)\nabla_\mu u^\mu \\ \frac{de}{d\tau} &= -(e+p)\,\frac{1}{\tau} \\ \frac{d(\tau e)}{d\tau} &= -p \end{aligned}$$

$$\begin{aligned} V &= \tau\,(\pi R_A^2 \Delta\eta) \\ dV &= d\tau\,(\pi R_A^2 \Delta\eta) \\ \frac{1}{V}\frac{dV}{d\tau} &= \frac{1}{\tau} \end{aligned}$$

- The energy per unit rapidity (τe) decreases due to the $p\,dV$ work in the longitudinal direction.

- The pressure is due to the thermal smearing of longitudinal velocities.

Viscosity and the Bjorken Expansion:

For a Bjorken expansion close to mid-rapidity: $v_z = z/t$

$$\begin{aligned} (\tau^{ij})_\eta &= -\eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta^{ij}\partial_l v^l\right) \\ (\tau^{ij})_\sigma &= -\sigma\delta^{ij}\left(\partial_l v^l\right) \end{aligned}$$

The only gradient is $\partial_z v^z = \frac{1}{V}\frac{dV}{d\tau} = \frac{1}{\tau}$

$$(\tau^{ij})_\eta = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} \quad x \quad & \quad y \quad & \quad z \quad \end{array} \\ \begin{pmatrix} +\frac{2}{3}\frac{\eta}{\tau} & & \\ & +\frac{2}{3}\frac{\eta}{\tau} & \\ & & -\frac{4}{3}\frac{\eta}{\tau} \end{pmatrix} \end{array} \quad \text{and} \quad (\tau^{ij})_\sigma = -\frac{\sigma}{\tau}\,\delta^{ij}$$

The stress energy tensor is:

$$\begin{aligned} T^{ij} &= T_o^{ij} + (\tau^{ij})_\eta + (\tau^{ij})_\sigma \\ &= \begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} + \begin{pmatrix} \frac{\frac{2}{3}\eta-\sigma}{\tau} & & \\ & \frac{\frac{2}{3}\eta-\sigma}{\tau} & \\ & & -\frac{\frac{4}{3}\eta+\sigma}{\tau} \end{pmatrix} \end{aligned}$$

- The Longitudinal Pressure is reduced by $\frac{4}{3}\eta/\tau$.
- The Transverse Pressure is increased by $\frac{2}{3}\eta/\tau$.

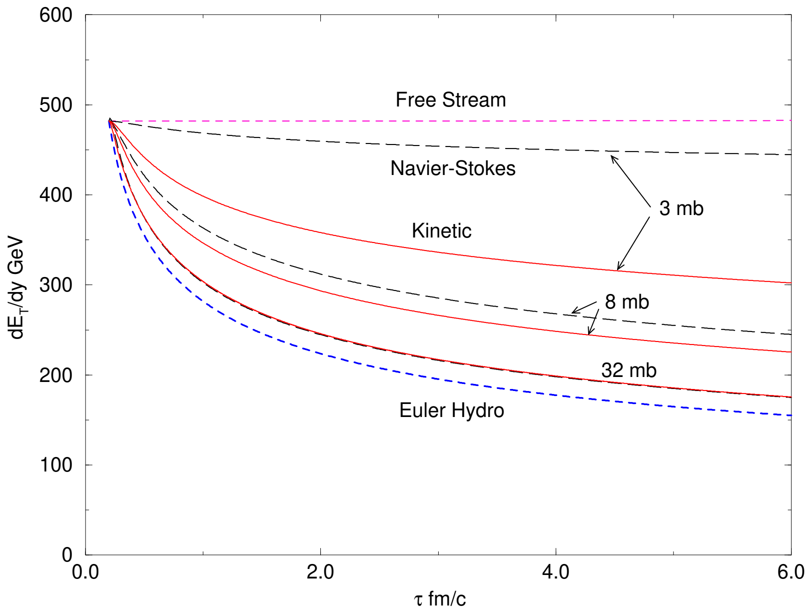
Now write down the equations of motion:

$$\underbrace{De = -(e+p)\nabla_\mu u^\mu}_{T\partial_\mu(su^\mu)} - \tau^{\mu\nu}\partial_\mu u_\nu \;\Rightarrow\; \underbrace{\frac{de}{d\tau} = -\frac{e+p}{\tau}}_{T\frac{d(\tau s)}{d\tau}} + \frac{\frac{4}{3}\eta + \sigma}{\tau}\frac{1}{\tau}$$

The equations of motion are:

$$\frac{d(\tau e)}{d\tau} = -\underbrace{\left(p - \frac{\frac{4}{3}\eta + \sigma}{\tau}\right)}_{T^{zz}} \;\Rightarrow\; T\frac{d(\tau s)}{d\tau} = \frac{\frac{4}{3}\eta + \sigma}{\tau}$$

- The longitudinal pressure is reduced in the longitudinal direction. The $p\,dV$ work is also reduced.
- The entropy/rapidity (i.e. τs) increases as a function of time as $\frac{\frac{4}{3}\eta+\sigma}{\tau}$



$$\frac{d(\tau e)}{d\tau} = -p + \frac{4}{3}\eta/\tau$$

- For Euler Hydro and ideal gas: $p = \frac{1}{3}\epsilon$, $\epsilon = \epsilon_0 \left(\frac{T}{T_0}\right)^4$

$$(\tau\epsilon) = (\tau_0\epsilon_0)\left(\frac{\tau_0}{\tau}\right)^{1/3} \quad \text{and} \quad T = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3}$$

  This has a simple explanation:

$$\frac{d(\tau s)}{d\tau} = 0 \text{ with } s \propto T^3 \text{ find } T \propto \frac{1}{\tau^{1/3}}$$

  $n \propto T^3$ and $\tau n = \frac{dn}{dy} = \text{Const}$

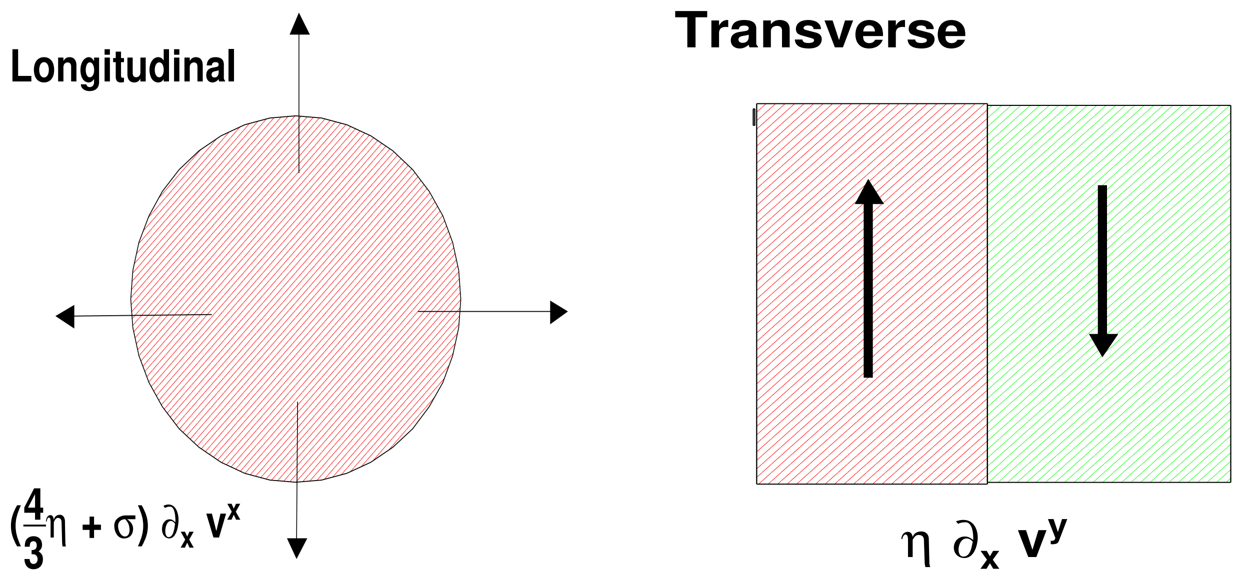
- For a free streaming gas: energy/rapidity is conserved $p = 0$.

$$(\tau\epsilon) = (\tau_0\epsilon_0) \quad \text{and} \quad T = T_0\left(\frac{\tau_0}{\tau}\right)^{1/4}$$

Temperature decrease slowly, $T \propto \left(\frac{1}{\tau}\right)^{1/3 \div 1/4}$

The particles per rapidity increases slowly, $\tau n \propto \tau^{0 \div 1/4}$

- η and σ appear in the combination $\frac{4}{3}\eta + \sigma$.



- It is impossible to construct a flow which is damped only by the the bulk viscosity σ
- This makes it hard to measure σ.

How valid is Hydro? How much Entropy is produced?

$$\frac{d(\tau s)}{d\tau} = 0 \qquad \text{(Ideal Case)}$$
$$\frac{d(\tau s)}{d\tau} = \frac{\frac{4}{3}\eta+\sigma}{\tau T} \qquad \text{(Viscous Case)}$$

For hydrodynamics to be valid, the entropy produced over the time scale of the system, τ , must be small compared to the total :

$$\tau \, \frac{\frac{4}{3}\eta + \sigma}{\tau T} \, \frac{1}{\tau s} \equiv \Gamma_s \frac{1}{\tau} \ll 1$$

where $\Gamma_s \equiv \frac{\frac{4}{3}\eta+\sigma}{e+p}$ is the *Sound Attenuation Length*. Note $(sT = e + p)$.

Estimates of Shear Viscosity:

$$\underbrace{T^{00} + T^{zz}}_{\text{indep. of Bag Const.}} = (e+p) + \eta \left\langle \partial^z v^z \right\rangle$$

$$\sim (e+p)\left[1 + \underbrace{\ell_{m.f.p.}}_{1/(n\sigma_0)} \left\langle \partial^z v^z \right\rangle\right]$$

Using $e + p \sim nT$, then find:

$$\Gamma_s \sim \ell_{m.f.p.} \quad \text{and} \quad \eta \sim (d.o.f.)\frac{T}{\sigma_0} \quad \text{and} \quad \frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \ll 1$$

Estimates of η for the QGP and Heavy Ion Collisions

Perturbative QCD – Arnold, Moore, Yaffe.

- $\eta \approx 86\, T^3 \frac{1}{g^4 \log(1/g)}$. Based upon kinetic theory of quarks and gluons.
  Estimate: (d.o.f) $= 36$ and $\sigma_0 \sim \frac{g^4}{T^2} 4$ then $\eta \sim 36\, \frac{T^3}{g^4}$.
  Set $\alpha_s \to 1/2$ and $\log(1/g) \to 1$

$$\left(\frac{\Gamma_s}{\tau}\right) \approx 0.18 \frac{1}{\tau T} \tag{1}$$

- Leading order. $\eta \approx 150\, T^3 \frac{1}{g^4}$ then

$$\left(\frac{\Gamma_s}{\tau}\right) \approx 0.4 \frac{1}{\tau T}$$

Strongly Coupled conformal N=4 SYM – Son, Starinets, Policastro

- No kinetic theory exists. Like most real liquids.

$$\left(\frac{\Gamma_s}{\tau}\right) = \frac{1}{3\pi} \frac{1}{\tau T} \approx 0.11 \frac{1}{\tau T} \tag{2}$$

Phenomenology – Molnar

Found could fit elliptic flow $v_2(p_T)$ only when

- $\frac{dN}{d\eta} = 1000$, $\sigma_0 = 10 \div 20$, and $\tau_o = 0.1$ fm.

$$\Gamma_s = 0.421 \frac{1}{n\sigma_0} \qquad \left(\frac{\Gamma_s}{\tau}\right) = 0.02 \div 0.04 \tag{3}$$

- Constant cross section. Independent of time!

Thermalization: How does Γ_s/τ evolve?

1. Bjorken Expansion | Scale Invariant Cross Section: $\sigma \sim \frac{\alpha_s^2}{T^2}$

- When Entropy is conserved: $T \sim \frac{1}{\tau^{1/3}}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \# \frac{1}{\tau^{2/3}}$$

- When $\frac{dE}{dy}$ is conserved (entropy is produced): $T \sim \frac{1}{\tau^{1/4}}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \# \frac{1}{\tau^{3/4}}$$

$\Longrightarrow$ rapid to rapid++ thermalization

2. Bjorken Expansion | Constant Cross Section: $\sigma = \sigma_0$

- When Entropy is Conserved: $\tau n \sim$ Const

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau n \sigma_0} \sim \text{Const}$$

- When $\frac{dE}{dy}$ is conserved (entropy is produced): $\tau n \sim \tau^{1/4}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau n \sigma_0} \sim \frac{1}{\tau^{1/4}}$$

$\Longrightarrow$ Constant to slow thermalization

Spherical Expansion | Scale Invariant Cross Section: $\sigma \sim \frac{\alpha_s^2}{T^2}$

- Entropy conservation: $(sV) \sim$ Const and $s \sim T^3$. Then $T \sim \frac{1}{\tau}$.

$$\frac{\Gamma_s}{\tau} \sim \frac{\#}{\tau T} \sim \text{Const}$$

- Energy conservation: $(eV) \sim$ Const and $e \sim T^4$ $T \sim \frac{1}{\tau^{3/4}}$

$\Longrightarrow$ Constant to slow thermalization

Spherical Expansion | Constant Cross Section: σ_0

- Entropy Conservation: $(nV) \sim$ Const. $n \sim \frac{1}{\tau^3}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau\, n\sigma_0} \sim \frac{\tau^2}{\sigma_0}$$

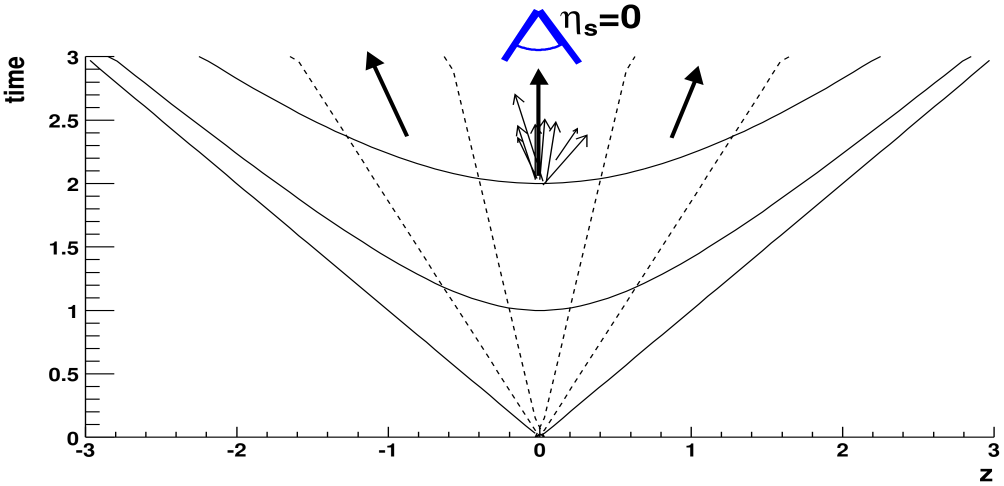
- Energy Conservation: $(eV) \sim$ Const. and $n \sim T^3 \sim 1\tau^{9/4}$

$$\frac{\Gamma_s}{\tau} \sim \frac{\ell_{m.f.p.}}{\tau} \sim \frac{1}{\tau\, n\sigma_0} \sim \frac{\tau^{5/4}}{\sigma_0}$$

$\Longrightarrow$ rapid++ to rapid breakup.

- Non-equilibrium processes (entropy production) do not seem to dramatically change the course of thermalization.
- The introduction of mass scales is of essential importance to the question of thermalization in heavy ion collisions.
- The problem is three dimensional after a $\tau \sim R_A$.

Thermal smearing



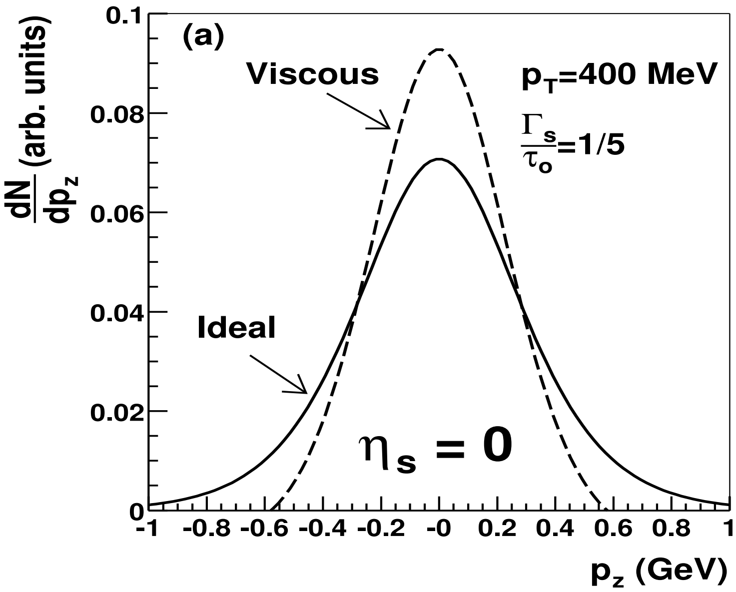
In equilibrium the thermal distribution is

$$f_0 = \frac{1}{e^{p_\alpha U^\alpha/T} - 1} = \frac{1}{e^{m_T \cosh(y-\eta_s)} - 1} \to \frac{1}{e^{E/T} - 1}$$

The effect of the viscosity is to reduce the longitudinal pressure.

$$T^{zz} = p - \frac{4}{3}\frac{\eta}{\tau} = \int d^3p \frac{p^z a p^z}{E}(f_0 + \delta f)$$

The situation is the following:



Want to calculate δf: Use the linearized Boltzmann equation

$$\frac{p^\mu}{E}\partial_\mu f_p = \int_{1,2,3} d\Gamma_{12\to 3p}\,(f_1 f_2 - f_3 f_p)$$

Linearize the Boltzmann equation:

- Substitute $f \to f^e + \delta f$ with $f^e_p = e^{-pu/T}$
- Keep first order in gradients.
- Use equilibrium: $f^e_1 f^e_2 = f^e_3 f^e_4$

$$\frac{p^\mu}{E}\partial_\mu f^e_p = \int_{1,2,3} d\Gamma_{12\to 3p}\, f^e_1 f^e_2 \left[\frac{\delta f_1}{f^e_1} + \frac{\delta f_2}{f^e_2} - \frac{\delta f_3}{f^e_3} - \frac{\delta f_4}{f^e_4}\right]$$

This is an integral equation for δf.

- δf is proportional to the strains: $\langle \nabla_\mu u_\nu \rangle, \nabla_\mu u^\mu, \nabla_\mu T$.
- δf is a scalar $\delta f \propto \chi(p) p^\mu p^\nu \langle \partial_\mu u_\nu \rangle$.
- If I restrict $f(p) = f_o(1 + \text{g(p)})$ where g(p) is a polynomial of degree less than two. Then the form is completely determined:

$$f = f_o(\frac{p\cdot u}{T})\left(1 + \frac{C}{T^3} p_\mu p_\nu \frac{\langle \partial^\mu u^\nu \rangle}{2}\right)$$

- This is sometimes called the first approximation

Refined analysis.

$$\frac{p^\mu}{E}\partial_\mu f_p^e = \int_{1,2,3} d\Gamma_{12\to 3p}\, f_1^e f_2^e \left[\frac{\delta f_1}{f_1^e} + \frac{\delta f_2}{f_2^e} - \frac{\delta f_3}{f_3^e} - \frac{\delta f_4}{f_4^e}\right]$$

Which gradients actually appear? $\partial_\mu = -u_\mu D + \nabla_\mu$

$$p^\mu \partial_\mu \left(e^{-pu/T}\right) = -f^e \Bigg[\underbrace{-(p\cdot u)\,(\frac{p}{T}\cdot Du)} - (p\cdot u)^2 D\left(\frac{1}{T}\right) + \frac{p^\mu p^\alpha}{T}\nabla_\mu u_\nu + \underbrace{(p\cdot u)\,(p\cdot\nabla\left(\frac{1}{T}\right))}\Bigg]$$

- Use ideal EOM $Du = -\frac{\nabla p}{e+p}$ then find

$$\underbrace{\cdots} \propto \frac{1}{T}\frac{\nabla p}{e+p} + \nabla\left(\frac{1}{T}\right) = \frac{n}{e+p}\nabla(\mu/T) = 0$$

- $D(1/T) \propto De$. Then use ideal EOM $De = -(e+p)\nabla_\mu u^\mu$

Put it all together:

$$p^\mu \partial_\mu f^e \quad = \quad -f^e \left[\underbrace{\left(-\frac{(p\cdot u)^2}{T} \frac{e+p}{T c_v} + \frac{1}{3} \frac{p\cdot\Delta\cdot p}{T} \right)}_{} \nabla_\mu u^\mu \right.$$

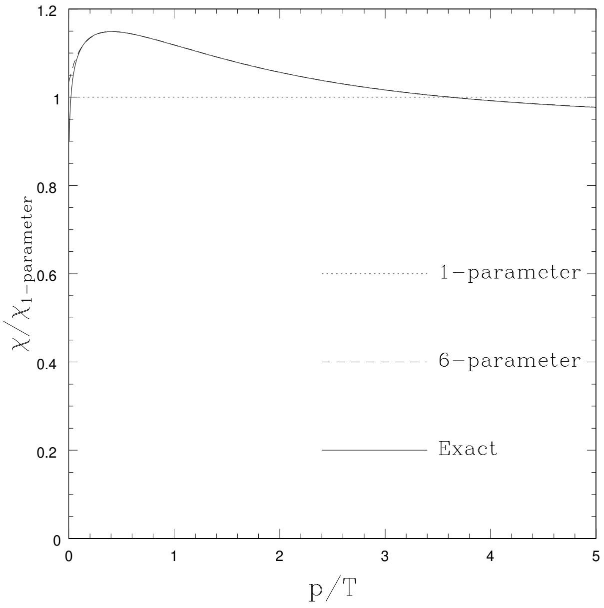
$$\left. + \frac{p^\mu p^\alpha}{T} \langle \nabla_\mu u_\alpha \rangle \right] = C[\delta f]$$

- The form of the viscous correction is not determined by symmetry alone.
  - $\nabla \left(\frac{1}{T}\right)$ Gets converted to $\nabla\left(\frac{\mu}{T}\right)$ using the ideal e.o.m. and the properties of the equilibrium distribution.
  - Microscopic manifestation of the Onsager relation.
- Look at the Bulk viscosity. For a massless ideal gas we have: $\epsilon \propto T^4$ and $T c_v = 4\, e$ and $e + p = \frac{4}{3} e$.

$$\underbrace{\cdots}_{} = 0$$

  - The bulk viscosity vanishes for a scale invariant ultra-relativistic gas. (Also vanishes for a non-relativistic Boltzmann gas)
- The form of the Shear Correction motivates the polynomial ansatz taken in the previous slide.

$$\delta f \propto f^e \frac{p_\mu p_\nu}{T} \langle \partial^\mu u^\nu \rangle$$



$$f = f_o(\tfrac{p\cdot u}{T})\left(1 + \frac{C}{T^3} p_\mu p_\nu \frac{\langle \partial^\mu u^\nu \rangle}{2}\right)$$

The constant $\frac{C}{T}$ is essentially the sound attenuation constant:

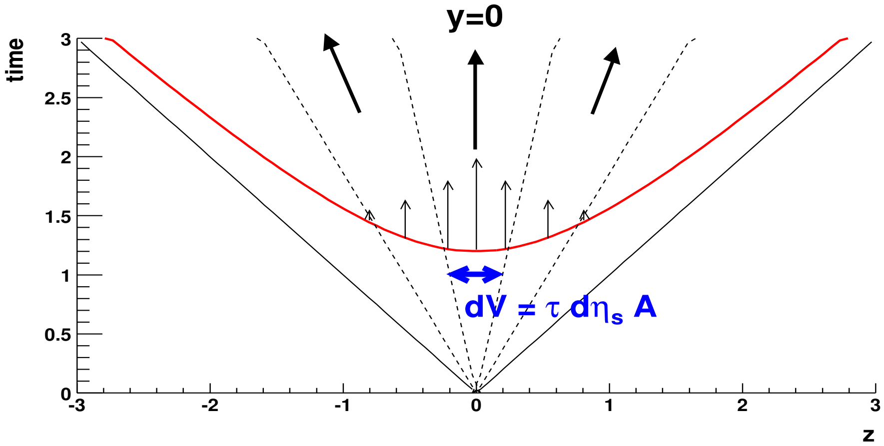
$$\begin{aligned} T^{\mu\nu} &= \int d^3p\, \frac{p^\mu p^\nu}{E}\, f \\ T_o^{\mu\nu} + T_{vis}^{\mu\nu} &= \int d^3p\, \frac{p^\mu p^\nu}{E}\, (f_o + \delta f) \end{aligned}$$

Then looking only at the viscous piece:

$$T_{vis}^{\mu\nu} = \eta \, \langle \partial^\mu u^\nu \rangle = \underbrace{\int d^3p\, \frac{p^\mu p^\nu}{E}\, f_o\, \frac{C}{T^3} p_\alpha p_\beta}\, \frac{\langle \nabla_\alpha u_\beta \rangle}{2}$$

- Classical Gas: C = $\frac{\eta}{s}$
- Massless Bose Gas: C = 0.96 $\frac{\eta}{s}$
- $\frac{\frac{3}{4}C}{T} \approx \Gamma_s$

Thermal Transverse Momentum Spectra at Mid rapidity:

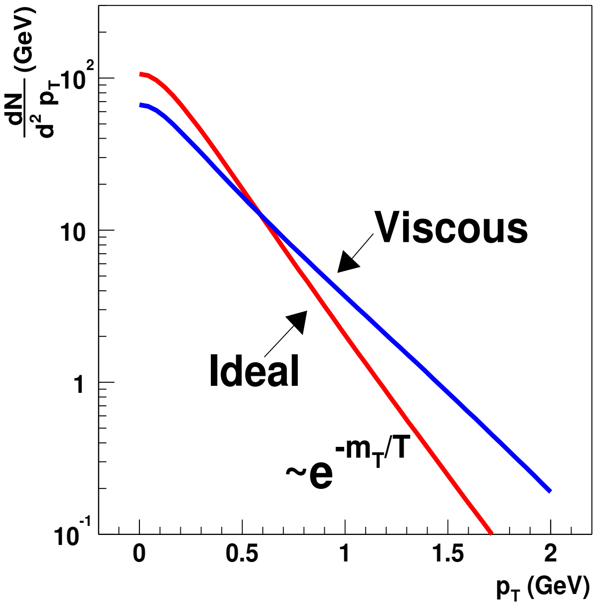


$$E\frac{d^3N}{d^3p} = \frac{d^2N}{d^2p_T\,dy} \quad = \quad \int \underbrace{p^\mu d\Sigma_\mu}_{E\,dV} \; f$$

Lets compute this integral:

$$\begin{aligned}
\frac{dN}{d^2p_T\,dy} &= \int dV \;\, m_T \cosh(\eta_s)\, e^{-pu/T} \\
&= \int A\,\tau d\eta_s \;\, m_T \cosh(\eta_s)\; e^{-\frac{m_T}{T}\cosh(\eta_s)} \\
&= (A\,\tau)\; m_T \;\; 2\,K_1\left(\frac{m_T}{T}\right)
\end{aligned}$$

Viscous corrections to p_T spectrum: Qualitative



The transverse pressure is larger with viscosity:

$$T_{zz} = p - \frac{4}{3}\frac{\eta}{\tau}$$

$$T_{xx} = T_{yy} = p + \frac{2}{3}\frac{\eta}{\tau}$$

Viscous corrections to p_T spectrum: Quantitative

$$dN_o + \delta\, dN = \int p^\mu d\Sigma_\mu\, f_o + \delta f$$

Want to compute $\frac{\delta\, dN}{dN_o}$:

$$\delta f = f_0 \Gamma_s \frac{p_\alpha}{T}\frac{p_\beta}{T}\Gamma_s \left\langle \nabla^\alpha u^\beta \right\rangle \sim f_0 \left(\frac{p_T}{T}\right)^2 \frac{2}{3}\frac{\Gamma_s}{\tau}$$

Now you can do these integrals:

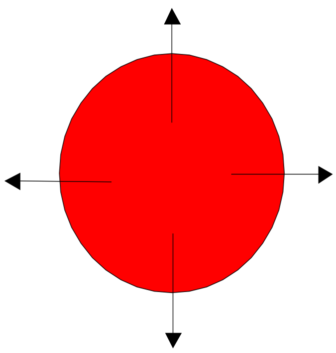
$$\frac{\delta\, dN}{dN_o} = \frac{\Gamma_s}{4\tau}\left\{\left(\frac{p_T}{T}\right)^2 - \left(\frac{m_T}{T}\right)^2 \frac{1}{2}\left(\frac{K_3(\frac{m_T}{T})}{K_1(\frac{m_T}{T})} - 1\right)\right\}$$

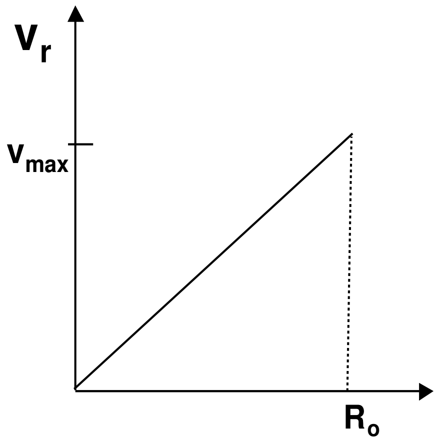
$$\rightarrow \frac{\Gamma_s}{4\tau}\left(\frac{p_T}{T}\right)^2$$

$$\frac{\delta\, dN}{dN_o} = \frac{\Gamma_s}{4\tau}\left(\frac{p_T}{T}\right)^2$$

- Viscous corrections grow with transverse momentum.
- When viscous corrections become of order one, then we are supposed to throw away hydrodynamics.
- Viscosity puts a bound on how high in p_T the hydrodynamics may be applied.
- For this room: $\Gamma_s/\tau \approx 10^4$ and $p_T^{max}/T = \sqrt{\frac{\tau}{\Gamma_s}} \approx 10^2$. $n \sim e^{-p/T}$ so you can forget about it.
- For heavy ion collisions: $T \approx 200\, MeV$ find $p_T^{\max} \approx 500\, MeV$. (Actually it is closer to 1.5 GeV)

Spectra with transverse expansion :





Compute $\langle \partial^\mu u^\nu \rangle$:

- Now Find two gradients: $\frac{1}{\tau}$ and $\frac{v_{max}}{R_o}$
- Compute the correction to the spectrum. Each term in $p_\mu p_\nu \langle \partial^\mu u^\nu \rangle$ gives a couple of Bessels.
- For large p_T

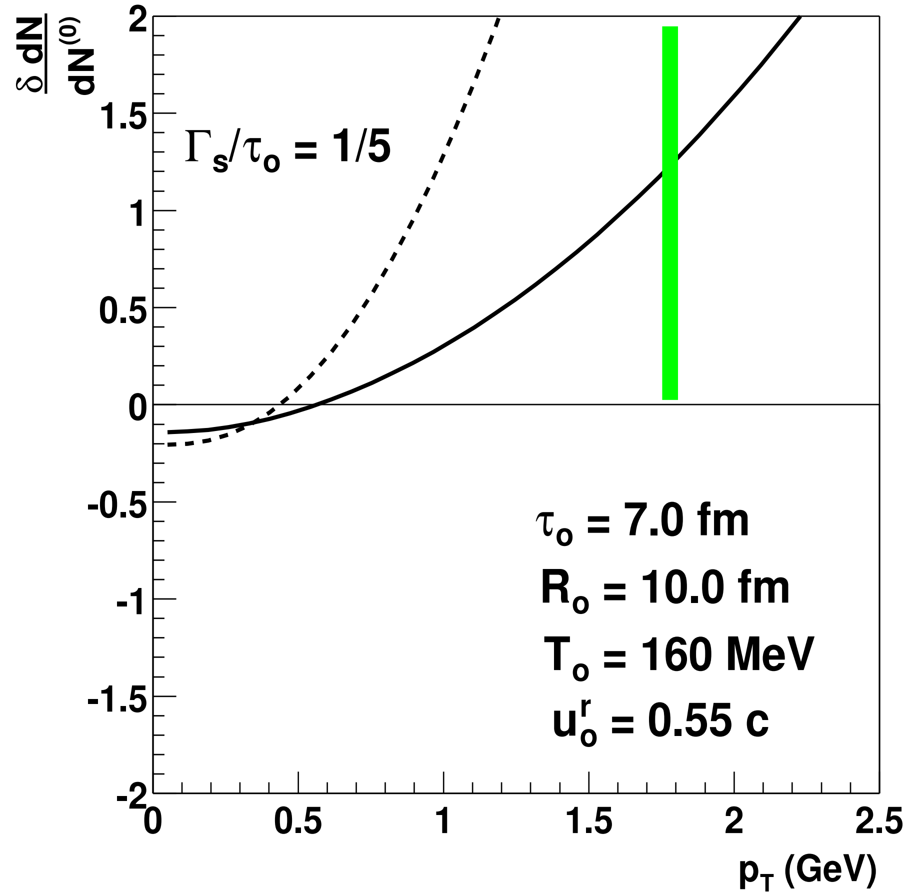
$$\frac{\delta\, dN}{dN_o} = \frac{\Gamma_s}{4\tau}\left(\frac{p_T}{T_{eff}}\right)^2\left(\frac{1}{\tau} - \frac{v_{max}}{R}\right)$$

where

$$T_{eff} = T\sqrt{\frac{1+v}{1-v}}$$

Two Mitigating Factors:

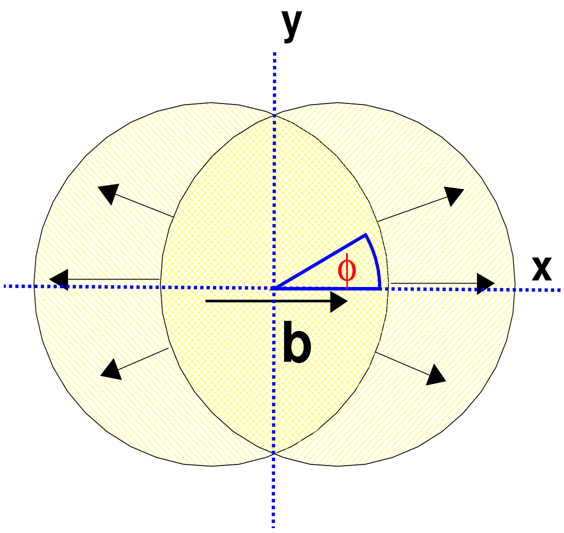
- The particles are blue shifted out to higher p_T
- The transverse flow alleviates the longitudinal shear.



$$\frac{\delta dN}{dN^{(0)}} \equiv \frac{\frac{dN^{(1)}}{p_T\, dp_T\, dy}}{\frac{dN^{(0)}}{p_T\, dp_T\, dy}}$$

The maximum possible p_T accessible to Hydrodynamics is ~ 1.8 GeV – A couple of times T_{eff} .

Elliptic Flow in Heavy Ion Collisions: Qualitative

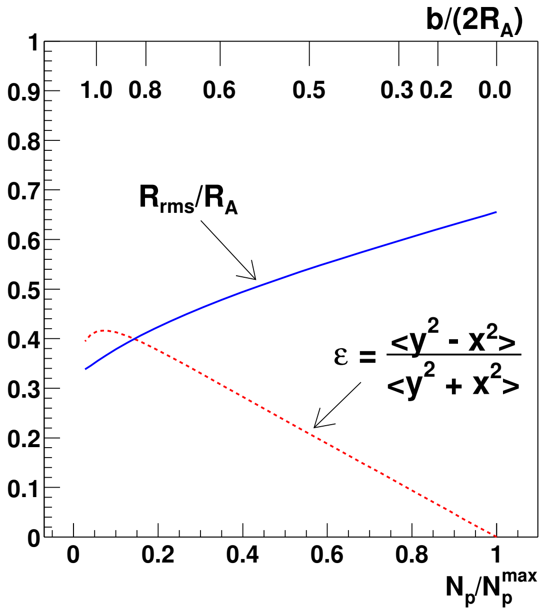


Measure the Anisotropy:

$$\frac{dN}{d\phi} = N(1+2\,v_2\cos(2\phi)+\cdots)$$

where $v_2 = \langle\cos(2\phi)\rangle$

Categorize the collision geometry:



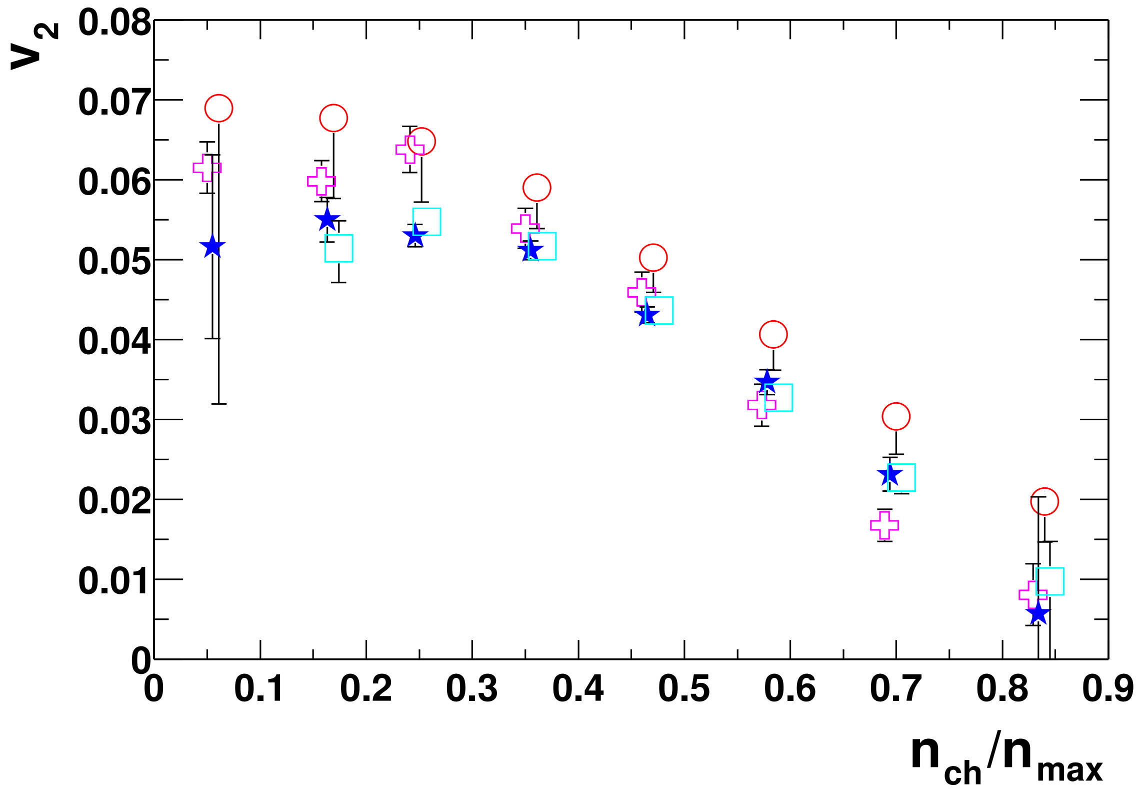
1. $N_p \equiv$ The number of $\underbrace{\text{participants}}$ participating nucleons.
2. $R_{\mathrm{rms}} \equiv \sqrt{\langle x^2 + y^2\rangle}$. The size of the collision zone.
3. $\epsilon \equiv$ The anisotropy of the initial geometry

Facts:

1. $\frac{dN}{dy} \propto N_p =$ the number of participants
2. $\epsilon \propto N_p =$ the number of participants nucleons.
3. Centrality $\approx \left(\frac{b}{2\,R_A}\right)^2$. Example $16 - 24\%$ central is $b \approx 7\,fm$

Observation of v_2 at RHIC!

Look at stars!



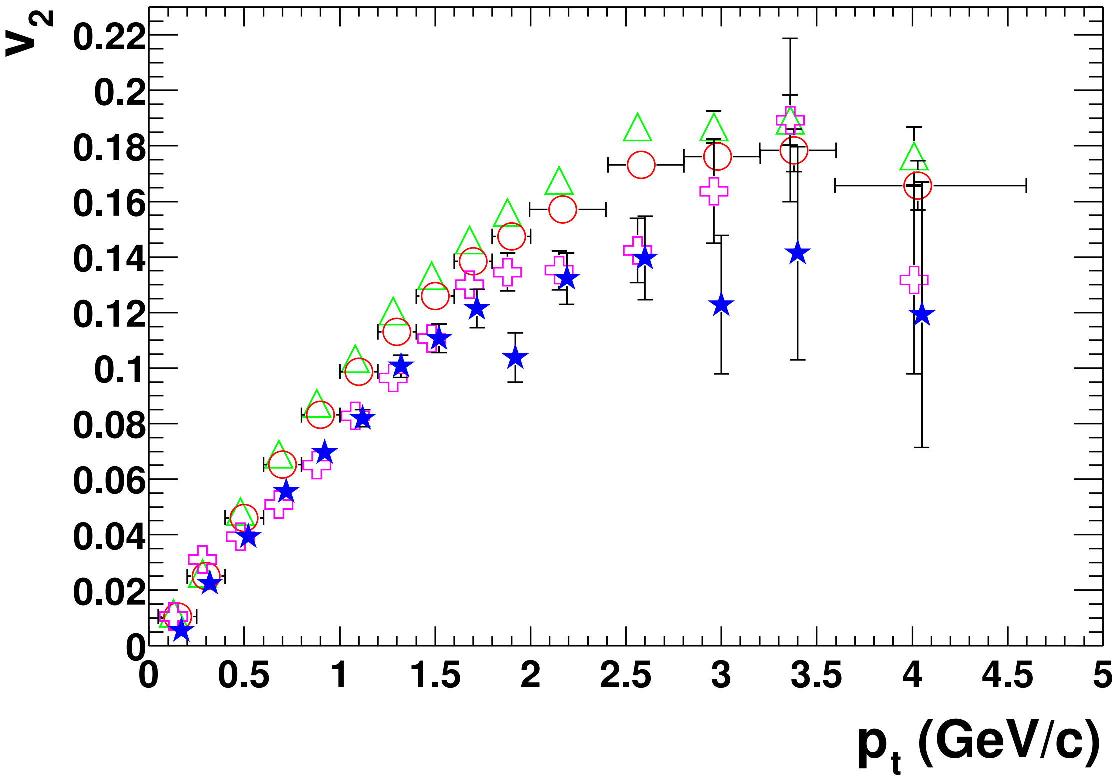
- If nothing changes as a function of centrality then expect: $v_2 \propto \epsilon$
- Up to corrections: , $v_2 \propto \epsilon$ in data

Also measure v_2 as a function of transverse momentum,

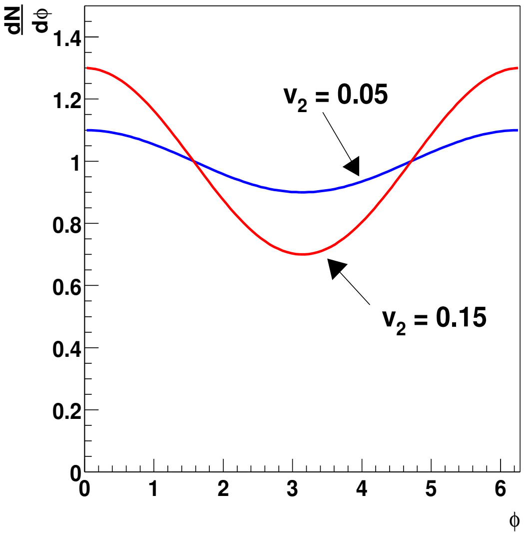
$$\frac{1}{p_T}\frac{dN}{dp_T\,d\phi} = \frac{1}{p_T}\frac{dN}{dp_T}\,(1 + 2\,v_2(p_T)\cos(2\,\phi)\cdots)$$

Then $v_2(p_T) \equiv \langle\cos(2\,\phi)\rangle_{p_T}$.

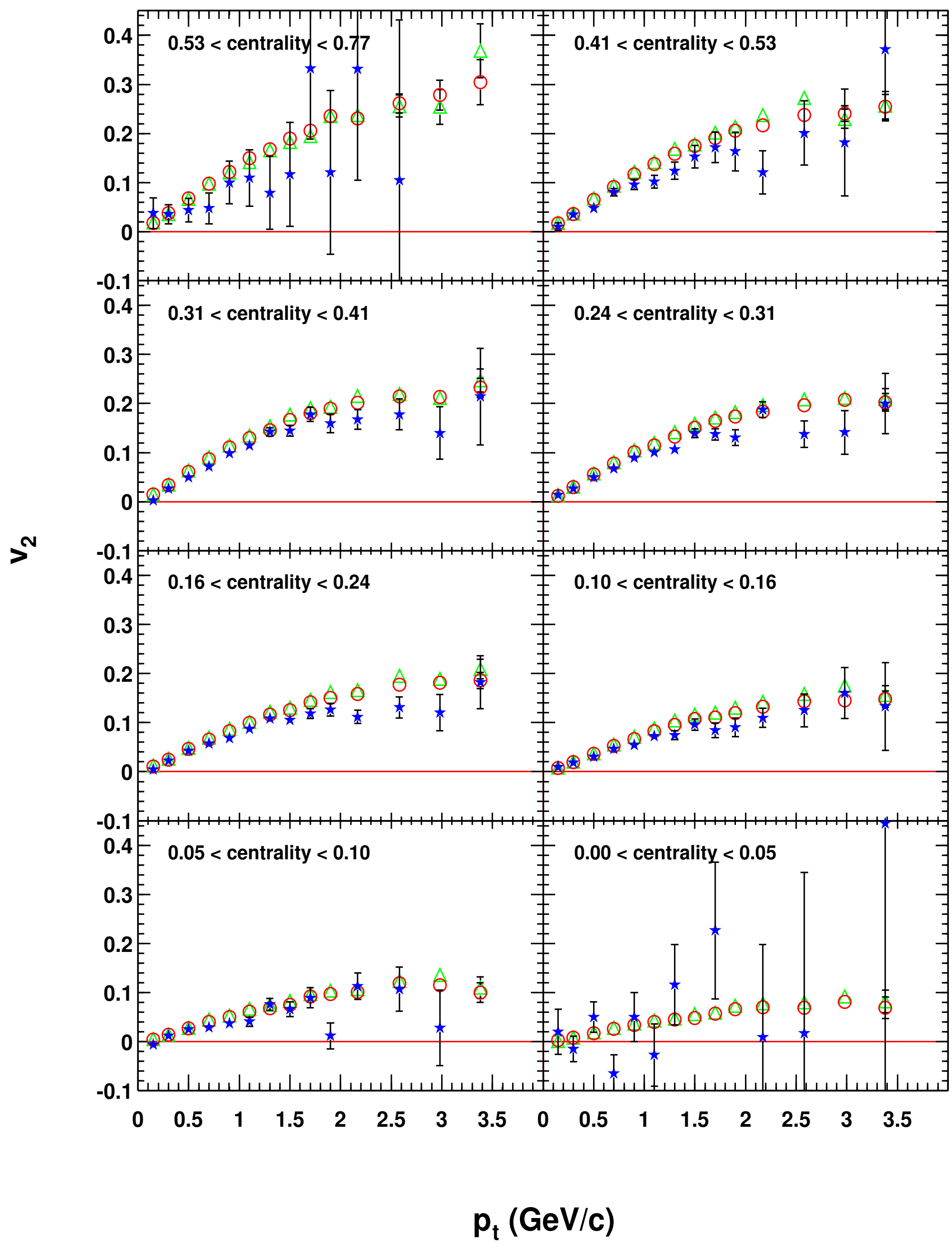
$b \approx 6\, fm$



- $v_2(p_T)$ increases as a function p_T until $p_T \approx 2.0$ GeV and then flattens at $v_2 \approx 0.15$
- v_2 is large even at $p_T \approx 4.0\, GeV$. There is a 1.8 to 1 asymmetry between x and y at $p_T = 3.0$ GeV.



Flow as function of centrality and p_T.



Basic Analysis of Elliptic Flow:

- Since ϵ is small we expect:

$$v_2 \propto \epsilon \propto 1 - N_p/N_p^{max}$$

- For a system with no other scales in the problem, the physics is independent of centrality. Expect

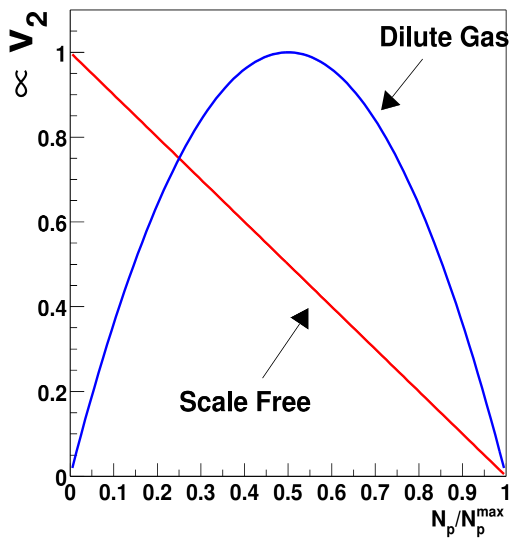
$$v_2 = \text{Cons}(1 - N_p/N_p^{max})$$

  Ideal hydrodynamics has no scales and the response is essentially trivially related to geometry.

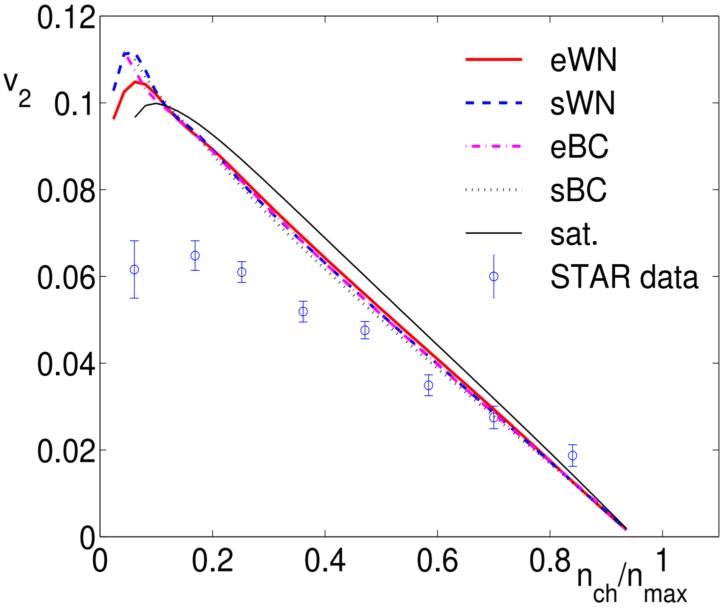
- For a dilute system (with constant cross sections) we expect collective response to be proportional to multiplicity $v_2 \propto \frac{dN}{dy} \propto N_p$.

$$v_2 \propto N_p(1 - N_p/N_p^{max})$$

- Viscous hydrodynamics is in-between these two cases.

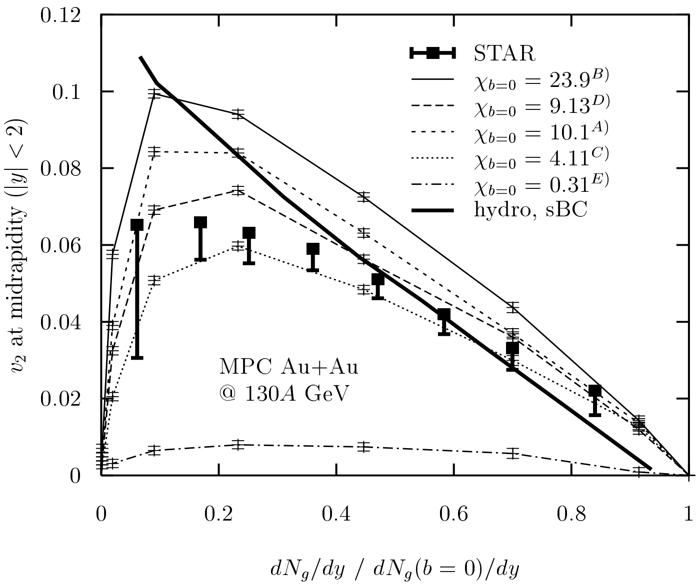


Ideal Hydrodynamics: (Peter Kolb and U. Heinz)



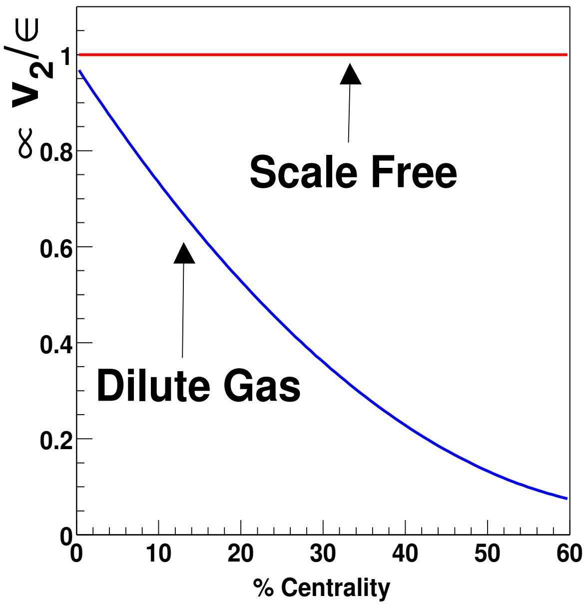
The solution is directly proportional to ϵ. There is no scale.

Solution to Boltzmann Equation: (Denes Molnar and M. Gyulassy)

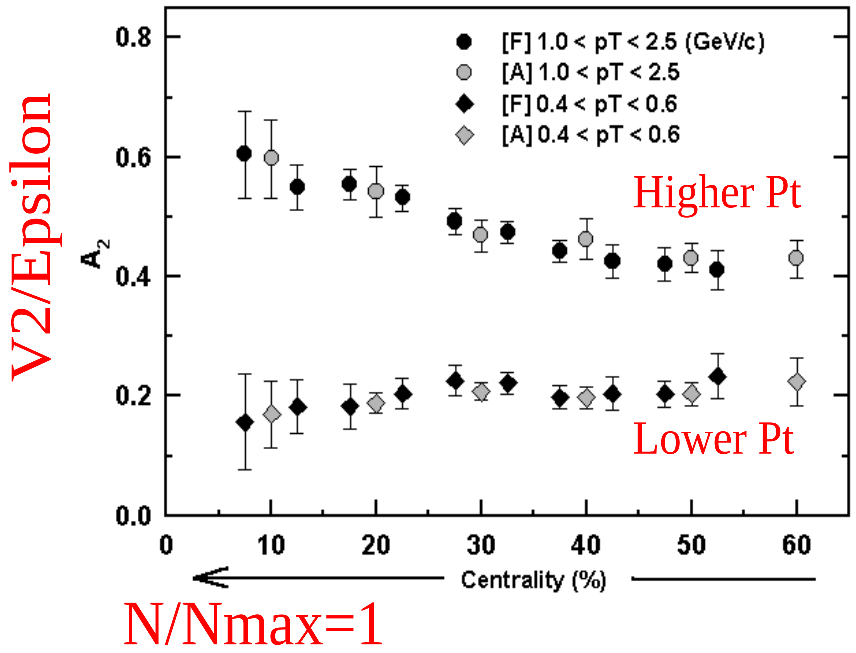


- For the Boltzmann equation, v_2 curves over in peripheral collisions.

Translation: Centrality $\approx (b/2R_A)^2$



Now let us look more differentially.

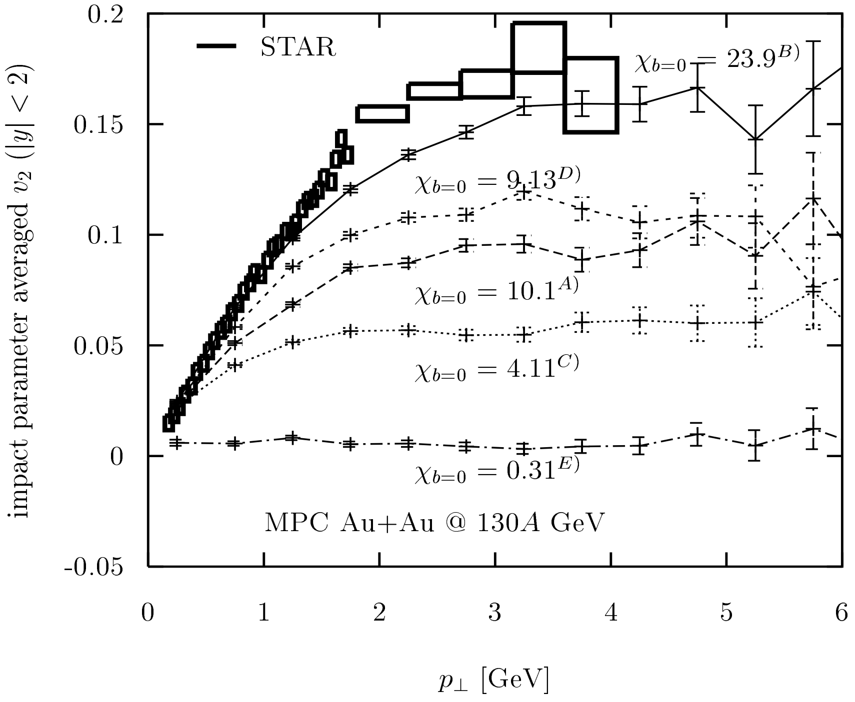


- At Lower Pt ≈ 0.6 GeV the response is directly proportional to ϵ
- At Higher Pt ≈ 1.4 GeV the effects of other scales come in.

This is naturally understood as a viscous effect.

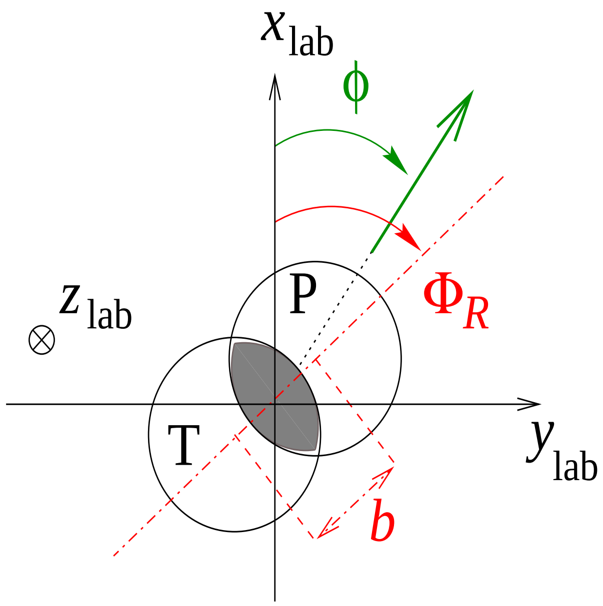
Solution of the Boltzmann Equation: Denes Molnar + M. Gyulassy

- Classical Massless Particles with Constant Cross Sections



- The Boltzmann equation predicted a flattening of v_2 at high p_T.
- The observed v_2 breaks down in a way consistently with viscous/Boltzmann effects.
- Need analytic understanding. Some understanding from viscous correction to the thermal distribution function.

Determination of Elliptic Flow: (N. Borghini and J.Y. Ollitrault)



How to determine Φ_R ?

- Two particles widely separated in phase space are not generally correlated.
- If there is flow then all particles are correlated with the reaction plane.
- Extract v_2 from inter-particle correlations.

Define: $\langle\rangle \equiv$ The average over all events

- The definition of v_2 is

$$\left\langle e^{2\,i(\phi-\Phi_R)} \right\rangle \equiv v_2$$

- The reaction plane is random

$$\left\langle e^{2\,i\Phi_R} \right\rangle = 0$$

Look at the two particle irreducible correlation and assume no correlations except with the reaction plane:

$$\begin{aligned}\left\langle e^{2\,i(\phi_1-\phi_2)}\right\rangle &= \left\langle e^{2\,i(\phi_1-\Phi_R)-2\,i(\phi_2-\Phi_R)}\right\rangle \\ &= \left\langle e^{2\,i(\phi_1-\Phi_R)}\right\rangle \left\langle e^{2\,i(\phi_2-\Phi_R}\right\rangle + O(1/M_{part}) \\ &= v_2^2 + O(1/M)\end{aligned}$$

- The error is $\delta v_2 \propto O(1/\sqrt{M})$:

Now look at the four particle irreducible correlation:

$$\begin{aligned}c_4 &\equiv \left\langle e^{2\,i(\phi_1+\phi_2-\phi_3-\phi_4)}\right\rangle \\ &- \left\langle e^{2\,i(\phi_1-\phi_3)}\right\rangle \left\langle e^{2\,i(\phi_2-\phi_4)}\right\rangle - \left\langle e^{2\,i(\phi_1-\phi_4)}\right\rangle \left\langle e^{2\,i(\phi_2-\phi_3)}\right\rangle\end{aligned}$$

If we assume that there is no correlation except with the reaction plane we have:

$$c_4 = v_2^4 - 2v_2^4 + O(1/M^3) = -v_2^4 + O(1/M^3)$$

- The error is $\delta v_2 \propto O(1/M^{3/4})$.
- Flow like correlations do not decrease as we look at higher multiparticle correlation functions: $(|c_n|)^{1/n} = v_2$
- The contributions of long range correlations goes from $1/\sqrt{M}$ to $1/M$ as we go to higher and higher cummulants: (Non-Flow$_n$)$^{1/n} = O(1/M^{1-1/n})$

Relation to Yang-Lee Zeros: (R.S. Bahlero, N. Borgihni, J.Y.Ollitrualt)

Is there a way to extract the limit: $\lim_{n\to\infty}(|c_n|)^{1/n} = v_2$?

The Advection Diffusion Equation:

Ideal Advection: Moving Along With the Stream

$$N^\mu = nu^\mu$$

Viscous Diffusion:

$$N^\mu = nu^\mu + J_D^\mu$$

For Navier Stokes:

$$J_D^\mu = \lambda(g^{\mu\nu} + u^\mu u^\nu)\partial_\nu n$$

In the rest frame of the fluid the conservation law becomes the ordinary diffusion equation:

$$\partial_\mu N^\mu = 0 \quad \rightarrow \quad \begin{cases} \partial_t n + \partial_x j = 0 \\ j = -\lambda \partial_x n \end{cases}$$

Problems:

- Two time derivatives make the initial value problem ill defined.
- Infinite propagation speeds for parabolic differential equations.
- A linear stability analysis shows that the small perturbations grow exponentially.
- Gets worse for Navier-Stokes equations.

The Relaxation Time Approximation:

In the rest frame, it takes a time τ_R for the current to relax. Make j a dynamic variable.

$$
\begin{aligned}
\partial_t n + \partial_x j &= 0 \\
\partial_t j &= -\frac{(j + \lambda \partial_x n)}{\tau_R}
\end{aligned}
$$

For an Arbitrary Frame:

$$
\begin{aligned}
\partial_\mu N^\mu &= 0 \\
D J_D^\mu &= -\frac{(J_D^\mu + \lambda \nabla^\mu n)}{\tau_R}
\end{aligned}
$$

where

$$
\begin{aligned}
N^\mu &= n u^\mu + J_D^\mu \\
D J_D^\mu &= u^\alpha \partial_\alpha J^\mu \\
\nabla^\mu &= (g^{\mu\nu} + u^\mu u^\nu)\partial_\nu
\end{aligned}
$$

Questions:

- What is the relation of the relaxation time to the microscopic correlation function.
- Is this theory causal.
- Can I solve this system.

- Long time physics should be insensitive to short timescales in the problem.

Basics of Diffusion

A particle undergoing a random walk with velocity v_{th} and collision time τ_R.

$$\Delta L^2 \sim (v_{th}\tau_R)^2 \frac{T_{\text{diff}}}{\tau_R}$$

The mean width of a diffusing Gaussian grows:

$$\Delta L^2(t) = 2\lambda(T_{\text{diff}})$$

Therefore we have

$$\lambda \sim v_{th}^2\tau_R \sim v_{th}\ell_{\text{mfp}}$$

Or

$$\frac{\lambda}{\tau_R} \sim v_{th}^2 \quad \text{and} \quad \frac{\lambda}{L} \sim v_{th}\frac{\ell_{\text{mfp}}}{L} \ll 1$$

For a Gaussian of length L with velocity $v \sim v_{th}$, a typical Advection time scale is:

$$T_{\text{adv}} \sim L/v_{\text{fluid}}$$

The Diffusion timescale is:

$$T_{\text{diff}} \sim L^2/\lambda$$

Therefore:

$$T_{\text{adv}} \ll T_{\text{diff}}$$

Relation to the correlation function:

From L. Kadanoff and P.C. Martin, Ann. Phys. 1965.

Then assume a small deviation from equilibrium n follows the same EOM.

$$\partial_t n(x,t) = \lambda \partial_x^2 n(x,t)$$

With the definitions

$$\langle n(t)n(0)\rangle_k \equiv \int dk e^{-ikx} \langle n(x,t)n(0,0)\rangle$$

we have

$$\partial_t \langle n(t)n(0)\rangle_k = -\lambda k^2 \langle n(t)n(0)\rangle_k$$

Then introduce the one sided Fourier transform in time.

$$(nn)^+_{kw} \equiv \int_0^\infty e^{iwt} \langle n(t)n(0)\rangle_k$$

This is related to the correlator by:

$$(nn)_{kw} \equiv \int_{-\infty}^{\infty} e^{iwt} \langle n(t)n(0)\rangle_k = 2Re(nn)^+_{kw}$$

Then (one-sided) Fourier transforming the EOM we find:

$$(nn)_{kw} = \frac{2\lambda k^2}{w^2 + (\lambda k^2)^2} \langle n(0)n(0)\rangle_k$$

Using the fluctuation dissipation theorem:

$$(nn)_{kw} = 2T \frac{\text{Im}\, \alpha(k,w)}{w}$$

Where

$$\alpha(k,w) = (i\theta(t)[n(t), n(0)])_{kw}$$

Thus we obtain:

$$\frac{\text{Im}\, \alpha(k,w)}{w} = \frac{2\lambda k^2}{w^2 + (\lambda k^2)^2} \frac{\langle n(0)n(0)\rangle_k}{T} \tag{4}$$

By taking limits we obtain the Kubo formula.

$$\frac{\langle n^2 \rangle}{T}\lambda = \lim_{w\to 0} \lim_{k\to 0} \frac{w}{k^2} \text{Im}\, \alpha(k,w) \tag{5}$$

λ is only determined by the Long time behavior of the correlation function

Werner-Israel Type Equation:

$$\begin{aligned}\partial_t n + \partial_x j &= 0 \\ \partial_t j &= -(j + \lambda \partial_x n)/\tau\end{aligned}$$

Combining equations we find:

$$\begin{aligned}\partial_t^2 n + \frac{\partial_t n}{\tau} + \frac{\lambda}{\tau}\partial_x^2 n &= 0 \\ \text{Initial Conditions} &= \begin{cases} n = n(0) \\ \partial_t n = 0 \end{cases}\end{aligned}$$

As before introduce the one sided Fourier transform and solve for $(nn)^+_{kw}$. We find

$$\frac{\mathrm{Im}\,\alpha(k,w)}{w} = \frac{2\,\lambda k^2}{w^2 + \tau^2(w^2 - \frac{\lambda}{\tau}k^2)^2} \frac{\langle n(0)n(0)\rangle_k}{T}$$

- The correlator is the same, as before for small frequencies
- At high frequencies the correlator is different.

The Werner-Israel Equation at short times

$$\partial_t^2 n + \frac{\partial_t n}{\tau} + \frac{\lambda}{\tau}\partial_x^2 n = 0$$

$$\text{Initial Conditions} = \begin{cases} n = n(0) \\ \partial_t n = 0 \end{cases}$$

At short times the correlator (after spatial Fourier transforming)

$$\partial_t^2 \left\langle n(t) n(0) \right\rangle_k \Big|_{t=0} - \left[\frac{\lambda}{\tau}\right] k^2 \left\langle n(0) n(0) \right\rangle_k = 0$$

- The short time behavior is determined by the combination $\frac{\lambda}{\tau}$
- $\frac{\lambda}{\tau}$ is of order a typical thermal velocity squared v_{th}^2

An expression for the quantity $\frac{\lambda}{\tau}$ is after using the Fluctuation Dissipation Theorem:

$$2T \frac{\text{Im}\alpha(k,w)}{w} = (nn)_{kw}$$

We find

$$\lim_{k\to 0} \int \frac{dw}{2\pi} w^2 2T \frac{\text{Im}\alpha(k,w)}{w} = k^2 \left[\frac{\lambda}{\tau}\right] \left\langle n^2 \right\rangle$$

There is a sum rule relating this moment to the time derivative of the correlator at the origin.

The Riemann-Characteristic Problem

Write the equations of motion in the following form:

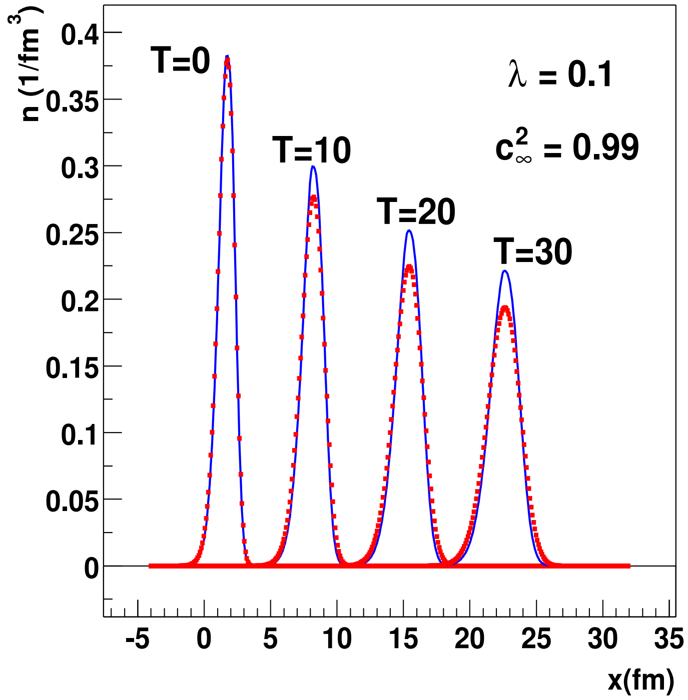
$$\left(\mathbf{B} \right) \begin{pmatrix} \partial_t n \\ \partial_t j \end{pmatrix} + \left(\mathbf{A} \right) \begin{pmatrix} \partial_x n \\ \partial_x j \end{pmatrix} = \begin{pmatrix} 0 \\ -j/\tau \end{pmatrix}$$

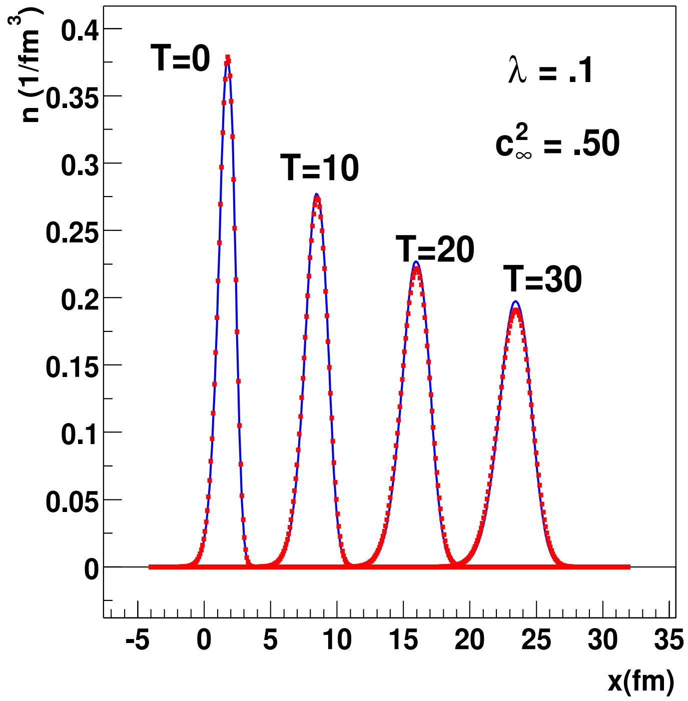
The eigen values/vectors of $\mathbf{B}^{-1}\mathbf{A}$ are the signal velocities of the equation of motion.

$$\begin{aligned} s_{\pm} &= \frac{v \pm c_\infty}{1 \pm v c_\infty} \\ c_\infty^2 &\equiv \frac{\lambda}{\tau} \end{aligned}$$

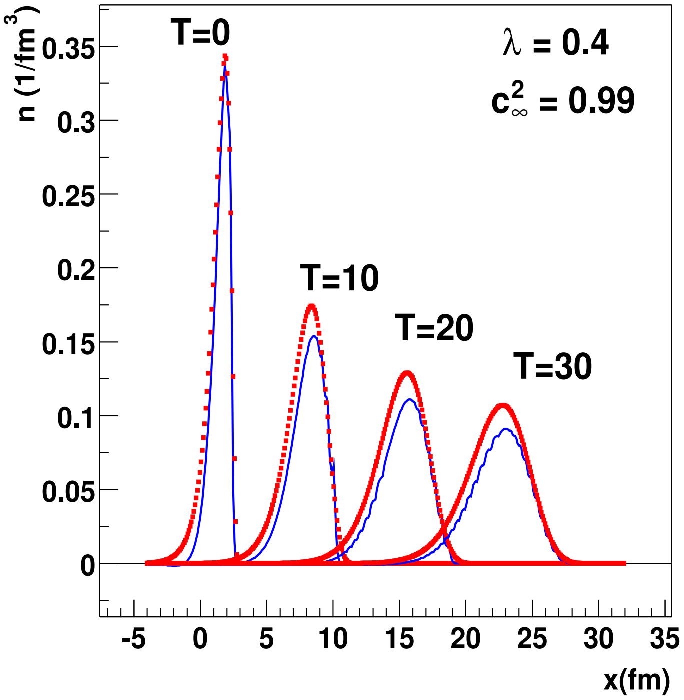
- Disturbances propagate with speed c_∞ in the local reference frame.
- c_∞ is of order the typical thermal velocity.
- If choose τ arbitrarily then the signals propagate faster than the speed of light.

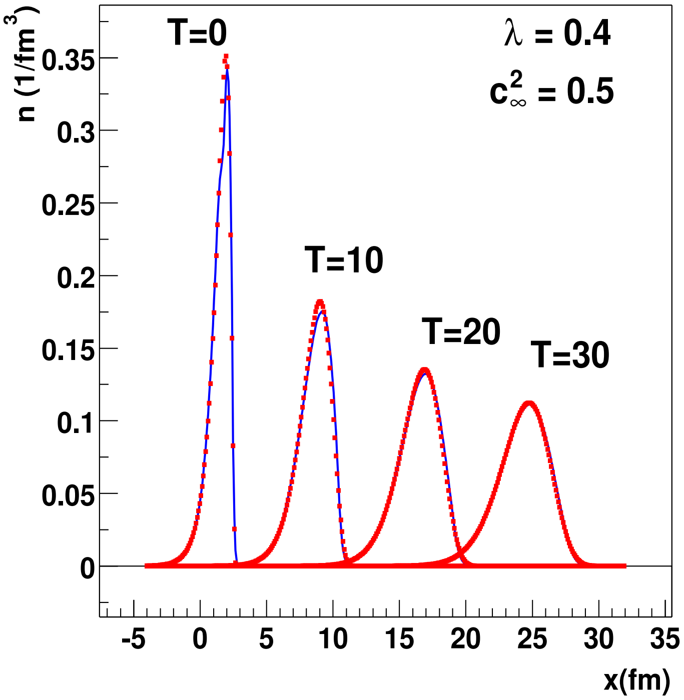
Small Diffusion Coefficient: v=0.85 !



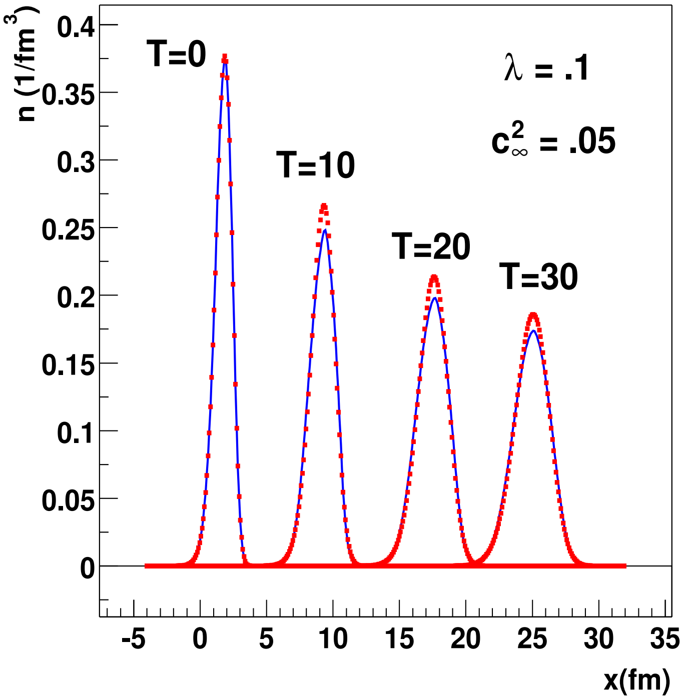


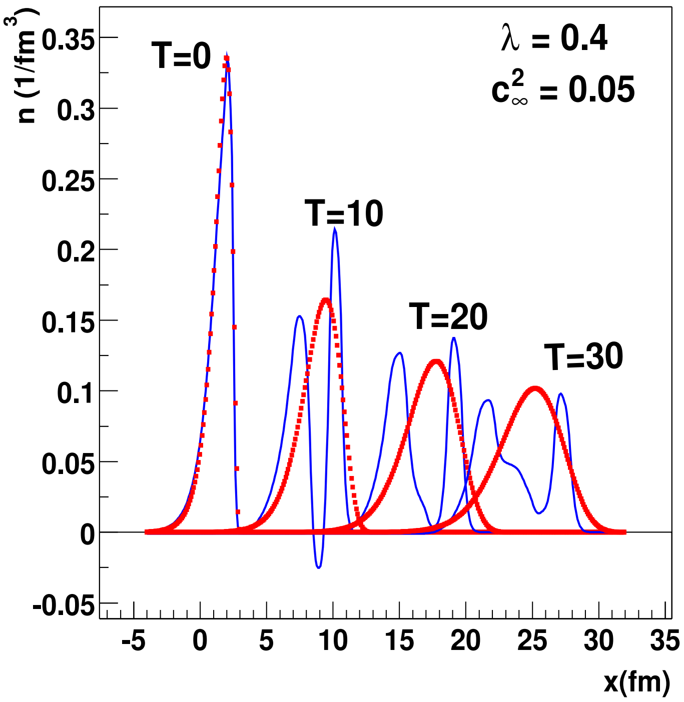
Large Diffusion Coefficient: v=0.85 !





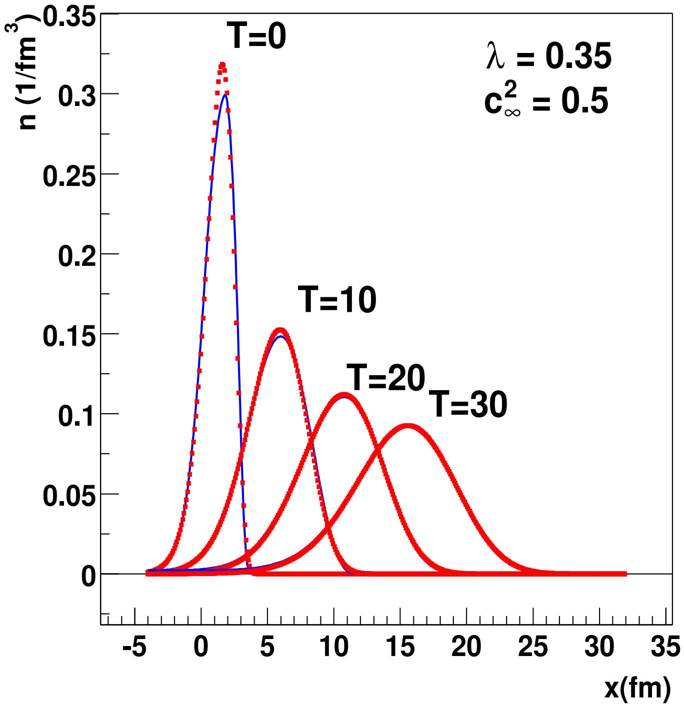
Problems when signal velocity $\rightarrow 0$





Goes crazy when outside regime of validity.

For more typical parameters v=.5, λ large



Conclusions

- Choose λ/τ to make the theory causal
- Compute λ/τ from short time response of correlation function
- For final results it does not matter what you choose/calculate for λ/τ.

  Only λ matters
- Always get ordinary Navier Stokes in the end.

Generalize this discussion to the Navier-Stokes Case:

Most naive approach:

$$T^{\mu\nu} = e\, u^{\mu} u^{\nu} + p\, \Delta^{\mu\nu} + \tau^{\mu\nu}$$

Where $\tau^{\mu\nu}$ vanishes in the rest frame : $\tau^{\mu\nu}\, u_{\mu} = 0$

and relaxes to equilibrium to over a time scale τ_R.

$$\Delta^{\mu\alpha}\Delta^{\nu\beta} D\tau_{\alpha\beta} = -\frac{\tau^{\mu\nu} + \eta\, \langle \nabla^{\mu} u^{\nu} \rangle}{\tau_R}$$

This is a pretty messy partial differential equation.

Let us look at the mode structure in one dimension.

The Riemann-Characteristic Problem

Write the equations of motion in the following form:

$$\left(\ \mathbf{B} \ \right) \begin{pmatrix} \partial_t e \\ \partial_t u^x \\ \partial_t \tau^{xx} \end{pmatrix} + \left(\ \mathbf{A} \ \right) \begin{pmatrix} \partial_x e \\ \partial_x u^x \\ \partial_x \tau^{xx} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\tau^{xx}/\tau_R \end{pmatrix}$$

The eigen values/vectors of $\mathbf{B}^{-1}\mathbf{A}$ are the signal velocities of the equation of motion.

$$\begin{aligned} s_0 &= v \\ s_\pm &= \frac{v \pm c_\infty}{1 \pm v c_\infty} \end{aligned}$$

where

$$\begin{aligned} c_\infty^2 &\equiv c_s^2 + \frac{1}{\tau_R} \frac{\frac{4}{3}\eta}{e + p - |\tau|} \\ |\tau| &\equiv \sqrt{\tau_{\mu\nu}\tau^{\mu\nu}} \end{aligned}$$

- c_∞ is of order the typical thermal velocity.
- If choose τ_R arbitrarily then the signals propagate faster than the speed of light.

$$
\begin{aligned}
c_\infty^2 &\equiv c_s^2 + \frac{1}{\tau_R} \frac{\frac{4}{3}\eta}{e+p-|\tau|} \\
|\tau| &\equiv \sqrt{\tau_{\mu\nu}\tau^{\mu\nu}}
\end{aligned}
$$

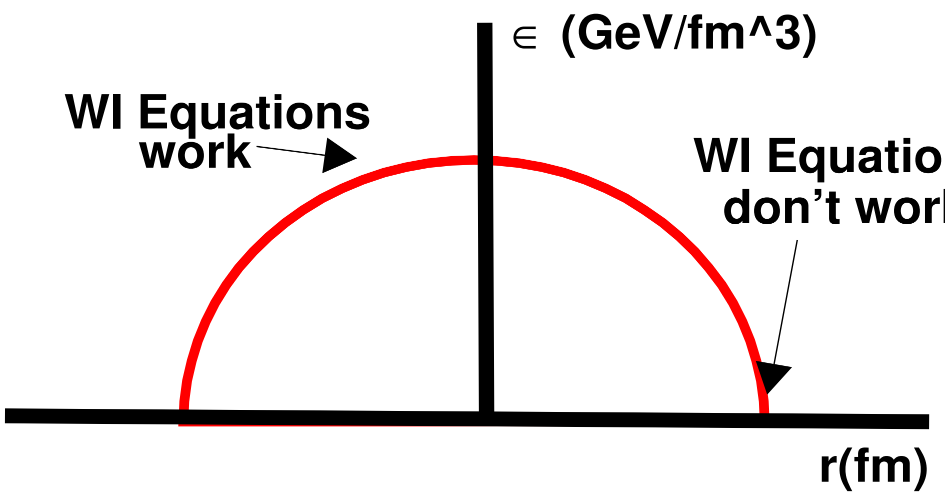
- For small $|\tau|$ can expand:

$$
c_\infty^2 \equiv c_s^2 + \frac{1}{\tau_R} \frac{\frac{4}{3}\eta}{e+p}
$$

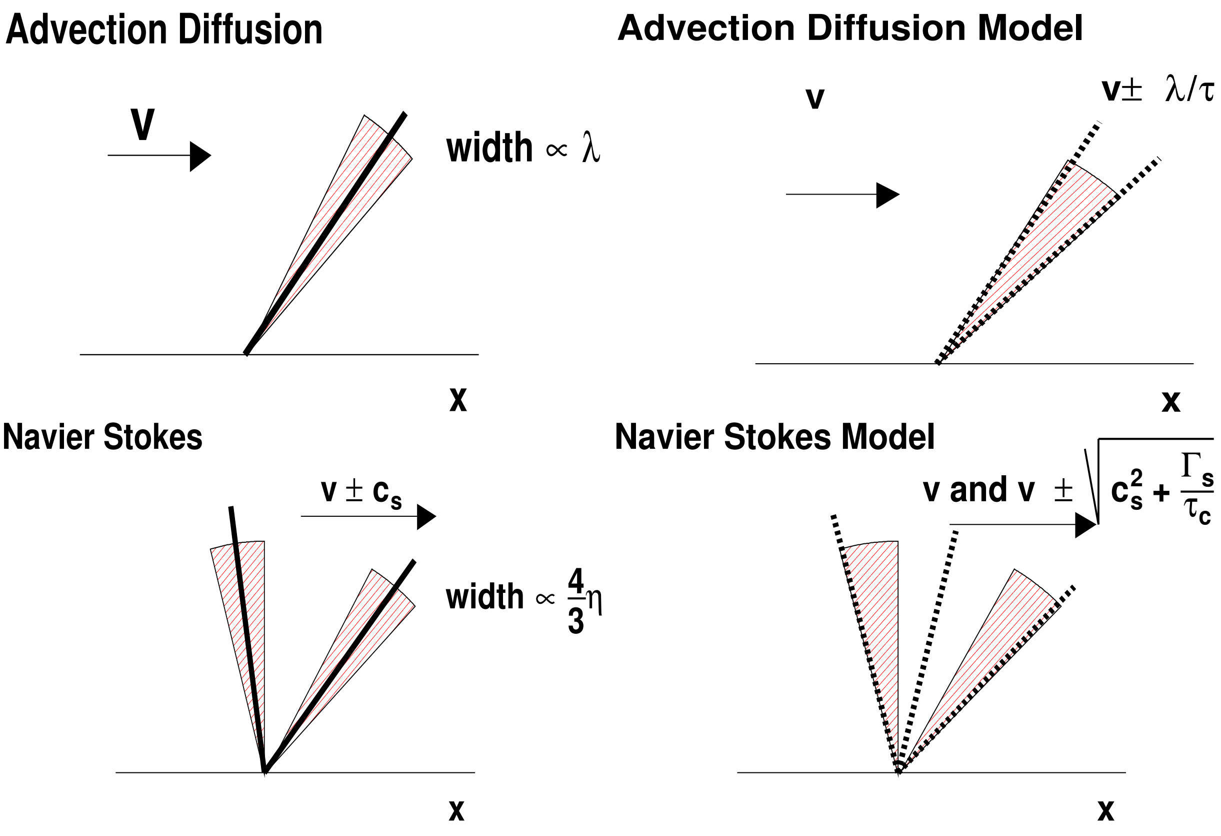
This is less than the speed of light.

- In truly non equilibrium situations $|\tau|$ is large.

Werner-Israel equations propagate faster than the speed of light.



Summary Of Mode Structure:



Remarkable observation: (L. Lindblom and R. Geroch)
The Navier Stokes Equation in the Eckart Frame can be written as a set of conservation/balance laws:

$$
\begin{aligned}
\partial_\mu(N^\mu) &= 0 \\
\partial_\mu(T^{\mu\nu}) &= 0 \\
\partial_\mu(A^{\mu\alpha\beta}) &= I^{\alpha\beta}
\end{aligned}
$$

where

$$
\begin{aligned}
N^\mu &= n u^\mu \\
T^{\mu\nu} &= e u^\mu u^\nu + p\Delta^{\mu\nu} + u^\mu q^\nu + u^\nu q^\mu + \tau^{\mu\nu} \\
A^{\mu\alpha\beta} &= 2T\Delta^{\mu(\alpha} u^{\beta)} \\
I^{\alpha\beta} &= -\frac{T}{\eta}\tau^{\alpha\beta} - \frac{2T}{3\sigma}\Delta^{\alpha\beta} - \frac{2T}{\kappa T}\left(q^\alpha u^\beta + q^\beta u^\alpha\right)
\end{aligned}
$$

Verify that (simple and key) :

$$\partial_\mu A^{\mu\alpha\beta} = T\left\langle \nabla^\alpha u^\beta \right\rangle + \text{Tensors of different symmetry}$$

Compare the terms in $\partial_\mu(A^{\mu\alpha\beta}) = I^{\alpha\beta}$ with the same symmetry:

$$T\left\langle \nabla^\alpha u^\beta \right\rangle = -\frac{T}{\eta}\tau^{\alpha\beta}$$

Now add a new contribution to $A^{\mu\alpha\beta}$:

$$B^{\mu\alpha\beta} = C_0\, u^\mu \tau^{\alpha\beta}$$

Note that:

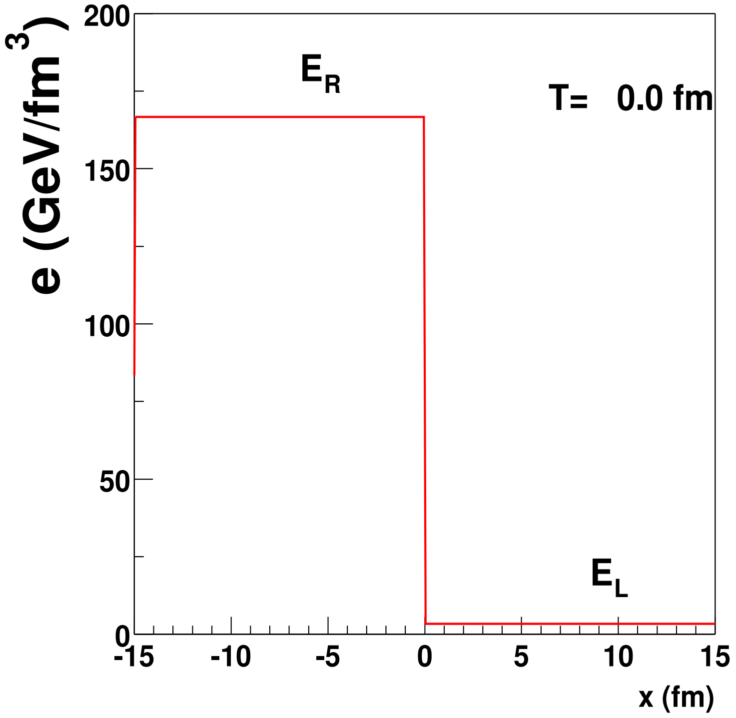
$$\partial_\mu B^{\mu\alpha\beta} \approx C_0\, u^\mu \partial_\mu \tau^{\alpha\beta} \approx C_0\, D\tau^{\alpha\beta}$$

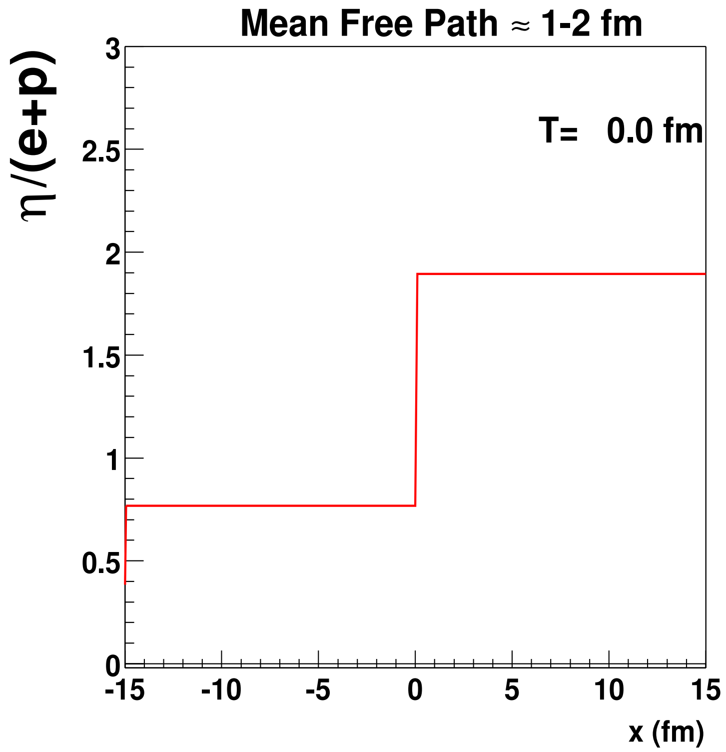
Then we see:

$$\begin{aligned} \partial_\mu (B^{\mu\alpha\beta} + A^{\mu\alpha\beta}) &= -\frac{T}{\eta}\tau^{\alpha\beta} + \text{Other symmetry} \\ C_0\, D\tau^{\alpha\beta} + \langle \nabla^\alpha u^\beta \rangle &= -\frac{T}{\eta}\tau^{\alpha\beta} + \text{Other symmetry} \end{aligned}$$

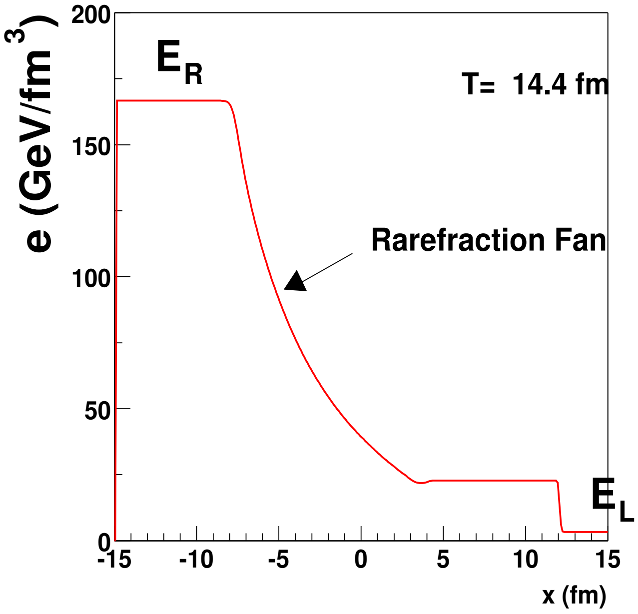
- This is the relaxation equation with a stiff source term.
- The relaxation equation can be formulated as a conservation law. (L. Linblom and R. Geroch)
- Can relate C_0 to τ_R and c_∞^2. It doesn't matter what you take for c_∞^2
- The relaxation models are not unique. Can add many such terms.

A Test Problem: Sod’s Problem

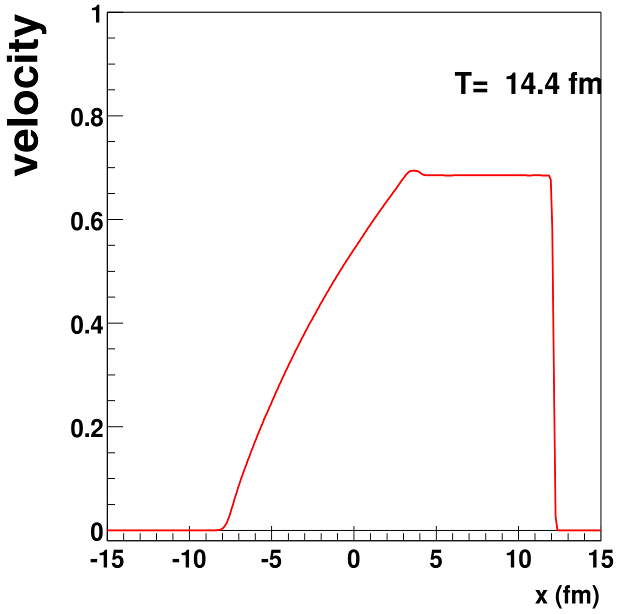




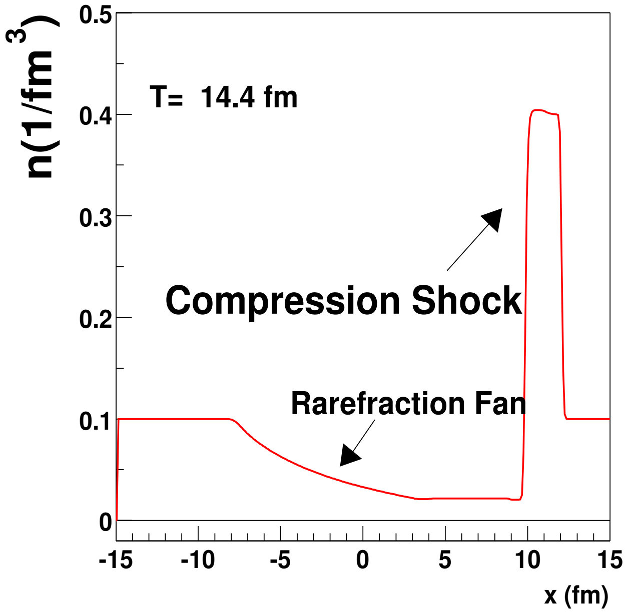
Euler Solution To Sod’s Problem:



E_R Rarefraction Fan E_L

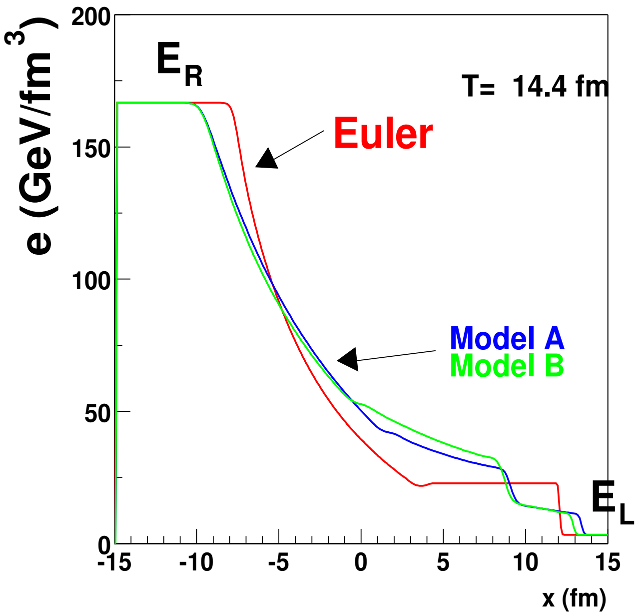


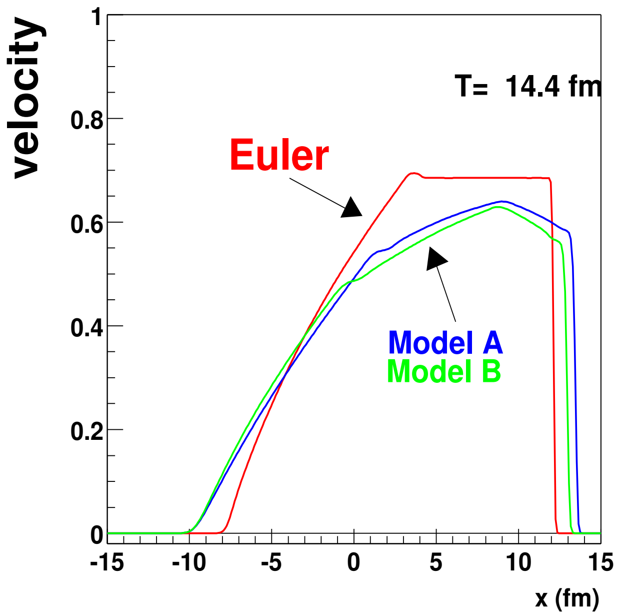
E_R E_L



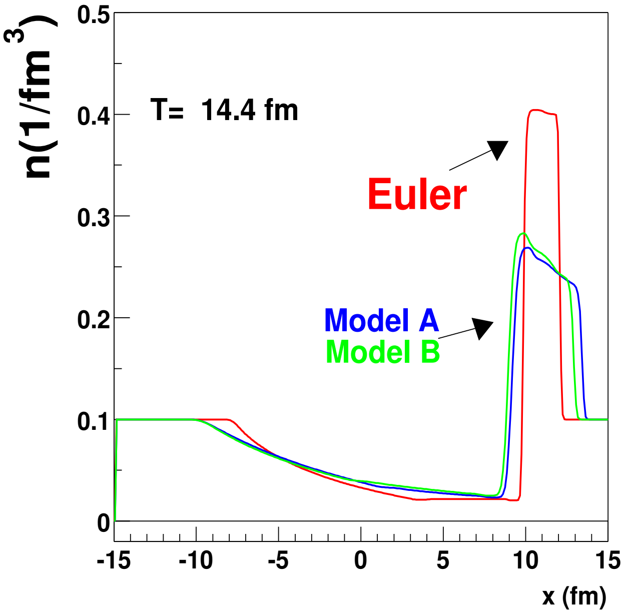
Viscous Solution To Sod's Problem

E_R E_L





E_R E_L



All Models are the SAME: