AVOIDING THE CHARACTERISTIC FUNCTION IN THE SUPERSYMMETRY METHOD

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Great Britain, Brunel (London), 1. December 2009

supported by SFB | TR12

Average over rotation invariant matrix ensembles are interesting for:

- disordered systems
- quantum chaos
- number theory
- matrix models in high energy physics
- etc.

EXAMPLE: HERMITIAN MATRICES

$$Z(x^-) \sim \int P(H) \frac{\prod_{n=1}^{q} \det(H - x^-_{n2})}{\prod_{m=1}^{p} \det(H - x^-_{m1})} d[H]$$

$$\sim \int P(E) \prod_{l=1}^{N} \frac{\prod_{n=1}^{q} (E_l - x^-_{n2})}{\prod_{m=1}^{p} (E_l - x^-_{m1})} |\Delta(E)|^2 d[E]$$

- energies with source variables and imaginary shift: $x^- = x - \imath\epsilon - J$
- eigenvalues of H: E_l

Introduction

Orthogonal polynomial method

- factorizable probability density P

$$P(E)=\prod_{j=1}^{N}P(E_j)\quad , \text{ with }\quad E=\mathrm{diag}\,(E_1,\ldots,E_N)$$

Arbitrary rotation invariant matrix ensembles

- non-factorizable ensembles
- proof of universality on the scale of the local mean level spacing in a mathematically rigorous way
- financial correlation matrices
- high-energy physics and quantum gravity: $P(H)\sim\exp(-\sum_j c_j\mathrm{tr}\,H^j)$
- fundamental mathematical question: supersymmetry for non-Gaussian ensembles

Historical remarks

First approaches of supersymmetry method

- for Gaussian matrix ensembles by Efetov et al. (80's)

$\longrightarrow$ corresponding superfunction is also Gaussian

Efetov-Wegner terms

- Efetov, Wegner, Constantinescu, and de Groote (80's)

$\longrightarrow$ Cauchy-like integral theorems

Supersymmetry method for arbitrary rotation invariant random matrix ensembles

Generalized Hubbard-Stratonovich transformation

- Guhr: Hermitian matrices (2006)
- Kieburg, Grönqvist, and Guhr: orthogonal and unitary-symplectic symmetry (2008/2009)

Superbosonization formula

Littelmann, Sommers, and Zirnbauer: all symmetry classes (2007/2008)

Comparison of both approaches

Kieburg, Sommers, and Guhr (2009)

Conventional Supersymmetry Method

$$\frac{1}{\det(H - x_{m1}^{-})} \sim \int \exp\left[\imath z_m^\dagger (H - x_{m1}^{-}) z_m\right] d[z_m]$$

vectors of ordinary variables z_m: $[z_{ma}, z_{mb}]_- = 0$

$$\det(H - x_{n2}^{-}) \sim \int \exp\left[\imath \eta_n^\dagger (H - x_{n2}^{-}) \eta_n\right] d[\eta_n]$$

vectors of Grassmann variables (G.v.) η_n: $[\eta_{na}, \eta_{nb}]_+ = 0$

The number of determinants determines the dimension of the superspace!

- interchange of Gaussian integrals with the average over the ordinary symmetric matrices
- performing the average

⇒ Fourier–transformation $\mathcal{F}$

$$P(H) \xrightarrow{\mathcal{F}} \mathcal{F}P(K)$$

- K contains z's and η's → see next transparency

CHARACTERISTIC FUNCTION

$$\mathcal{F}P(K) = \int P(H) \exp(\imath \operatorname{tr} HK) d[H]$$

inherits the rotation invariance from P

Here the characteristic function enters the calculation!

Wishart–like matrices

- $N \times N$–dyadic ordinary matrix:

$$K = \sum_{m=1}^{p} z_m z_m^\dagger - \sum_{n=1}^{q} \eta_n \eta_n^\dagger = V^\dagger V$$

- $(p+q) \times (p+q)$–supermatrix:

$$B = VV^\dagger$$

- $(p+q) \times N$ rectangular supermatrix V

Crucial identity

$$\operatorname{tr} K^m = \operatorname{Str} B^m$$

for all $m \in \mathbb{N}$
$\longrightarrow$ relation between **all** invariants in ordinary space and superspace

Duality relation

$$\mathcal{F}P(K) = \mathcal{F}P_0(\operatorname{tr} K, \operatorname{tr} K^2, \ldots) = \mathcal{F}P_0(\operatorname{Str} B, \operatorname{Str} B^2, \ldots) = \Phi(B)$$

The characteristic function of P can not be removed anymore!

Fourth Step: Fourier–transformation in superspace

Exchange of the integral over B by an integral over symmetric supermatrices

$$\Phi(B) \overset{?}{\longrightarrow} \Phi(\rho)$$

Heuristic approach

$$\Phi(B) = \int \Phi(\rho)\delta(\rho - B)d[\rho]$$

Two approaches for the step "**?**"

- Generalized Hubbard–Stratonovich transformation

$$\Phi(B) = \int\int \Phi(\rho)\exp[\imath \mathrm{Str}\,\sigma(\rho - B)]d[\rho]d[\sigma]$$

- Superbosonization formula

Fourth Step: Fourier–transformation in superspace

$$Q(\sigma) \sim \int \Phi(\rho) \exp(-\imath \mathrm{Str}\, \rho\sigma) d[\rho]$$

The Fourier-transform Q is not P!

We have to perform two Fourier–transformations and have no closed expressions $Q[P]$ and $\Phi[P]$ for arbitrary P!

Map from ordinary space to superspace such that

$$Z(x^-) \sim \int Q(\sigma)\mathrm{Sdet}^{-N}(\sigma - x^-)d[\sigma]$$

or in Fourier space

$$\begin{aligned} Z(x^-) &\sim \int \Phi(\rho) I(\rho) \exp(-\imath \mathrm{Str}\, \rho x^-) d[\rho] \\ &\sim \int \Phi(\rho)\mathrm{Sdet}^{N-p+q}\rho \exp(-\imath \mathrm{Str}\, \rho x^-) d[\rho] \end{aligned}$$

- Q probability density in superspace
- Φ is "characteristic function" of P **and** Q

Is there an explicit expression for Φ and Q?

$$\begin{aligned} \Phi &= \Phi[P] \\ Q &= Q[P] \end{aligned}$$

Alternative Supersymmetry Method

Outcome with the new approach

Explicit integral representation of Q and Φ

$$
\begin{aligned}
Q(\sigma) &\sim \int P\left(\begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix}\right) d[H,W] \\
\Phi(\rho) &= \int Q(\sigma)\exp(\imath \mathrm{Str}\,\rho\sigma) d[\sigma]
\end{aligned}
$$

P is an arbitrary rotation invariant probability density!

$p \leq N + q$

- H: $N - p + q$ dimensional ordinary hermitian matrix
- σ and ρ: (p/q) dimensional hermitian supermatrix
- W: $(N - p + q) \times (p + q)$ rectangular supermatrix

Cauchy-like integral theorem for rectangular matrices

Rotation invariance

- W: $l \times (k+k)$ dimensional rectangular supermatrix
- f: zero boundary condition at infinity
- $$f(W, W^\dagger) = f(WU, U^\dagger W^\dagger)$$
  for all $U \in \mathrm{U}(k/k)$

Cauchy-like integral theorem

$$\int f(W, W^\dagger) d[W] \sim f(0)$$

Wegner[83]; Constantinescu, de Groote[89]; Kieburg, Kohler, Guhr[08]

Cauchy-like integral theorem for symmetric matrices

Rotation invariance

- σ: (k/k) dimensional hermitian supermatrix
- f: zero boundary condition at infinity
- $$f(\sigma) = f(U\sigma U^\dagger)$$

  for all $U \in \mathrm{U}(k/k)$

Cauchy-like integral theorem

$$\int f(\sigma) d[\sigma] \sim f(0)$$

Wegner[83]; Constantinescu, de Groote[89]; Kieburg, Kohler, Guhr[08]

First Step: Enlarging the matrix

Consider the simplest example $p = q = k$

$$Z(x^-) \sim \int P(H) \prod_{n=1}^{k} \frac{\det(H - x_{n2}^-)}{\det(H - x_{n1}^-)} d[H]$$

$$\sim \int P\left(\begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix}\right) \prod_{n=1}^{k} \frac{\det(H - x_{n2}^-)}{\det(H - x_{n1}^-)} d[H, W, \sigma]$$

- σ: (k/k) dimensional hermitian supermatrix
- W: $N \times (k + k)$ dimensional rectangular supermatrix

Replacing the determinants by Gaussian integrals (**The same procedure as in the original approach!**)

$$
\begin{aligned}
Z(x^-) &\sim \int P\left(\begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix}\right) \exp\left(\imath \operatorname{tr} H V^\dagger V\right) \\
&\times \exp(-\imath \operatorname{Str} x^- V V^\dagger) d[H, W, \sigma, V] \\
&\sim \int P\left(\begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix}\right) \exp\left(\imath \operatorname{Str} \begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix} \begin{bmatrix} V^\dagger V & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\times \exp(-\imath \operatorname{Str} x^- V V^\dagger) d[H, W, \sigma, V]
\end{aligned}
$$

***P* is rotation invariant!**

Fourier–transformation in the $(N+k/k)$ **dimensional superspace**

$$\widehat{\mathcal{F}}P\left(\begin{bmatrix}\widetilde{H} & \widetilde{W}\\ \widetilde{W}^\dagger & \rho\end{bmatrix}\right) = \int d[H,W,\sigma]P\left(\begin{bmatrix}H & W\\ W^\dagger & \sigma\end{bmatrix}\right) \times \exp\left(\imath\mathrm{Str}\begin{bmatrix}H & W\\ W^\dagger & \sigma\end{bmatrix}\begin{bmatrix}\widetilde{H} & \widetilde{W}\\ \widetilde{W}^\dagger & \rho\end{bmatrix}\right)$$

$\widehat{\mathcal{F}}P$ **inherits the rotation invariance!**

$$Z(x^-) \sim \int \widehat{\mathcal{F}}P\left(\begin{bmatrix}V^\dagger V & 0\\ 0 & 0\end{bmatrix}\right)\exp(-\imath\mathrm{Str}\,x^- VV^\dagger)d[V]$$

Rotation invariance yields

$$\widehat{\mathcal{F}}P\left(\begin{bmatrix} V^\dagger V & 0 \\ 0 & 0 \end{bmatrix}\right) = \widehat{\mathcal{F}}P\left(\begin{bmatrix} 0 & 0 \\ 0 & VV^\dagger \end{bmatrix}\right)$$

due to

$$\mathrm{Str}\begin{bmatrix} V^\dagger V & 0 \\ 0 & 0 \end{bmatrix}^m = \mathrm{tr}\,(V^\dagger V)^m = \mathrm{Str}\,(VV^\dagger)^m = \mathrm{Str}\begin{bmatrix} 0 & 0 \\ 0 & VV^\dagger \end{bmatrix}^m$$

We have not changed the function as in the original approach!

$\widehat{\mathcal{F}}P$ is characteristic function of P

$$Z(x^-) \sim \int P\left(\begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix}\right) \exp\left(\imath \mathrm{Str} \begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & VV^\dagger \end{bmatrix}\right)$$
$$\times \exp(-\imath \mathrm{Str}\, x^- VV^\dagger) d[H, W, \sigma, V]$$

$$\sim \int P\left(\begin{bmatrix} H & W \\ W^\dagger & \sigma \end{bmatrix}\right) \mathrm{Sdet}^{-N}(\sigma - x^-) d[H, W, \sigma]$$

$$\overset{!}{\sim} \int Q(\sigma) \mathrm{Sdet}^{-N}(\sigma - x^-) d[\sigma]$$

We have removed the characteristic function!

Further simplifications

(I) $P(H) = P_0(\operatorname{tr} H, \operatorname{tr} H^2)$

- due to Cauchy-like integrals

$$Q(\sigma) \quad \sim \quad \int P\left(\begin{bmatrix} H & 0 \\ 0 & \sigma \end{bmatrix}\right) d[H]$$

- the same result as in Guhr (2006)

(II) Also true for:

bosonic eigenvalues of σ = fermionic eigenvalues of σ

(I) or (II) + P factorizes

$\Rightarrow Q(\sigma) = P(\sigma)$

HERMITIAN MATRIX ENSEMBLE

- supergroup integrals are known

$$Q(\sigma) \sim \int P\left(\begin{bmatrix} H & 0 \\ 0 & \sigma \end{bmatrix}\right) \mathrm{Sdet}\left(\sigma \otimes \mathbf{1} - \mathbf{1} \otimes H\right) d[H] + \text{b.t.}$$

- Q is up to the boundary terms (Efetov-Wegner terms) “b.t.” the generating function Z.

Advantage in the case $N \to \infty$

Consider the large N behavior of

$$\int Q(\sigma)\mathrm{Sdet}^{-N}(\sigma - x^-)d[\sigma]$$

for $p = q = k$ on the scale of the local mean level spacing

SADDLE POINT APPROXIMATION

- assume $\ln Q(\sigma)$ behaves for large N as a polynomial

$\Rightarrow$ every bosonic eigenvalue of the saddle point σ_0 is also a fermionic eigenvalue of σ_0

- other saddle points go with $1/N^\delta$, $\delta > 0$

$\Rightarrow$

$$Q(\sigma) \sim \int P\left(\begin{bmatrix} H & 0 \\ 0 & \sigma \end{bmatrix}\right) d[H]$$

- for factorizing $P \Rightarrow Q(\sigma) = P(\sigma)$

$\Rightarrow$ Q is approximately P for large N

Summary

New supersymmetry approach

- an alternative mapping to superspace; avoiding an explicit calculation of the characteristic function
- explicit integral expressions of the superfunctions Q and Φ
- another one–matrix solution for restrictions of the dimensions N, p and q [conventional solutions are Bunder et al. (2007) and Kieburg et al. (2009)]

This result is also true for

- arbitrary number of characteristic polynomials
- real symmetric and Hermitian self-dual matrices
- Wishart matrix ensembles, also for supersymmetric Wishart matrices
- circular ensembles

Thank you for your attention!