“SUPERSYMMETRY WITHOUT SUPERSYMMETRY”: A NEW APPROACH TO DERIVE DETERMINANTAL AND PFAFFIAN STRUCTURES

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Average over rotation invariant matrix ensembles are interesting for:

- disordered systems
- quantum chaos
- number theory
- matrix models in high energy physics
- Weyl’s character formula
- etc.
**Setting**

\[
Z(\kappa, \lambda) \sim \int P(H) \frac{\prod_{n=1}^{k_2} \det(H - \kappa_{n2}) \prod_{m=1}^{l_2} \det(H^\dagger - \lambda_{m2})}{\prod_{n=1}^{k_1} \det(H - \kappa_{n1}) \prod_{m=1}^{l_1} \det(H^\dagger - \lambda_{m1})} d[H]
\]

+ factorizable probability density \( P \)

\[
P(E) = \prod_{j=1}^{N} P(E_j) \quad \text{, with } E = \text{diag} (E_1, \ldots, E_N)
\]

**H has not to be symmetric!**
**First Example: Hermitian Matrices**  

\[ Z_N(\kappa) = \int_{\text{Herm}(N)} P(H) \prod_{j=1}^{k} \frac{\det(H - \kappa j_2)}{\det(H - \kappa j_1)} d[H] \]

\[ \sim \int \prod_{a=1}^{N} \left[ P(E_a) \prod_{j=1}^{k} \frac{E_a - \kappa j_2}{E_a - \kappa j_1} \right] |\Delta_N(E)|^2 d[E] \]

It is well known that this generating function exhibits a determinantal structure!

Baik, Deift, Strahov (2003); Grönqvist, Guhr, Kohler (2004); Borodin, Strahov (2005); Guhr (2006)
FIRST EXAMPLE: HERMITIAN MATRICES Herm (N)

ALGEBRAIC REARRANGEMENT

Multiplying

(I) = \prod_{a=1}^{N} \prod_{j=1}^{k} \frac{E_a - \kappa j_2}{E_a - \kappa j_1} \Delta_N(E)

by

(II) = \sqrt{\text{Ber}_{k/k}(\kappa)} \sim \det \left[ \begin{array}{c} 1 \\ \kappa a_1 - \kappa b_2 \end{array} \right]

yields

(I)(II) = \sqrt{\text{Ber}_{k/k+N}(\kappa, E)} \sim \det \left[ \begin{array}{cc} 1 & 1 \\ \kappa a_1 - \kappa b_2 & \kappa a_1 - E_b \\ \kappa b_2^{a-1} & E_b^{a-1} \end{array} \right]

The variables $\kappa a_2$ and $E_b$ are the new fermionic eigenvalues!
**First example: Hermitian matrices** \( \text{Herm} (N) \)

\[
Z_N(\kappa) \sim \int \prod_{a=1}^{N} P(E_a) \sqrt{\frac{\text{Ber}_{k/k+N}(\kappa, E)}{\text{Ber}_{k/k}(\kappa)}} \Delta_N(E) d[E]
\]

**Determinantal structure**

\[
Z_N(\kappa) \sim \frac{1}{\sqrt{\text{Ber}_{k/k}(\kappa)}}
\]

\[
\times \det \begin{bmatrix}
\frac{1}{\kappa a_1 - \kappa b_2} & \int \frac{P(E)E^{b-1}}{\kappa a_1 - E} dE \\
\kappa b_2^{a-1} & \int P(E)E^{a+b-2} dE 
\end{bmatrix}
\]
**First Example: Hermitian Matrices \( \text{Herm} (N) \)**

**Leibniz Expansion of the Determinant**

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det [A - BD^{-1} C]
\]

yields a \( k \times k \) determinant

**Comparing the Determinant Kernel for \( k = 1 \)**

\[
Z_N(\kappa) \sim \frac{1}{\sqrt{\text{Ber}_{k/k}(\kappa)}} \det \left[ \frac{Z_N(\kappa a_1, \kappa b_2)}{\kappa a_1 - \kappa b_2} \right]
\]

We have not used the explicit structure of \( P(E) \)!

→ These structures have to be true for other matrix ensembles!
**SECOND EXAMPLE: SPECIAL ORTHOGONAL GROUP SO(2N)**

\[ Z_{2N}(\kappa) = \int_{\text{SO}(2N)} \prod_{j=1}^{k} \frac{\det(O - \kappa j_2)}{\det(O - \kappa j_1)} d\mu(O) \]

\[ \sim \quad \text{Sdet}^{-N\kappa} \int \prod_{a=1}^{N} \left[ g(E_a) \prod_{j=1}^{k} \frac{E_a - \tilde{\kappa}_j}{E_a - \tilde{\kappa}_j} \right] |\Delta_N(E)|^2 d[E] \]

with

\[ g(E_a) \sim \frac{\Theta(E_a - 1)\Theta(1 - E_a)}{\sqrt{1 - E_a^2}} \quad ; \quad \tilde{\kappa}_{ji} = \frac{\kappa_{ji} + \kappa_{ji}^{-1}}{2} \]

**Interesting for mathematicians: Weyl–type character formula**
Huckleberry, Püttmann, Zirnbauer (2005)
Second Example: Special Orthogonal Group SO(2N)

Algebraic Rearrangement

Multiplying

\[ (I) = \prod_{a=1}^{N} \prod_{j=1}^{k} \frac{E_a - \tilde{\kappa}_j^2}{E_a - \tilde{\kappa}_j^1} \Delta_N(E) \]

by

\[ (II) = \sqrt{\text{Ber}_{k/k}(\tilde{\kappa})} \]

yields

\[ (I)(II) = \sqrt{\text{Ber}_{k/k+N}(\tilde{\kappa}, E)} \sim \det \begin{bmatrix} 1 & 1 \\ \tilde{\kappa}_{a1} - \tilde{\kappa}_{b2} & \tilde{\kappa}_{a1} - E_b \\ \tilde{\kappa}_{b2} & E_b \\ & & ^{a-1} \end{bmatrix} \]

The variables \( \tilde{\kappa}_{a2} \) and \( E_b \) are the new fermionic eigenvalues!
SECOND EXAMPLE: SPECIAL ORTHOGONAL GROUP SO($2N$)

$$Z_{2N}(\kappa) \sim \text{Sdet}^{-N\kappa} \int \prod_{a=1}^{N} g(E_a) \sqrt{\text{Ber}_{k/k+N}(\tilde{\kappa}, E)} \Delta_N(E) d[E]$$

DETERMINANTAL STRUCTURE

$$Z_{2N}(\kappa) \sim \frac{\text{Sdet}^{-N\kappa}}{\sqrt{\text{Ber}_{k/k}(\tilde{\kappa})}} \times \text{det} \begin{bmatrix} 1 & \int \frac{g(E) E^{b-1}}{\tilde{\kappa}_{a1} - E} dE \\ \tilde{\kappa}_{a1} - \tilde{\kappa}_{b2} & \int g(E) E^{a+b-2} dE \\ \tilde{\kappa}_{b2}^{a-1} & \end{bmatrix}$$
SECOND EXAMPLE: SPECIAL ORTHOGONAL GROUP SO(2N)

**Leibniz Expansion of the Determinant**

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det[A - BD^{-1}C]
\]

**Comparing the Determinant Kernel for \( k = 1 \)**

\[
Z_{2N}(\tilde{\kappa}) \sim \frac{1}{\sqrt{\text{Ber}_{k/k}(\tilde{\kappa})}} \det \left[ \frac{Z_{2N}(\tilde{\kappa}_a, \tilde{\kappa}_b)}{\tilde{\kappa}_a - \tilde{\kappa}_b} \right]
\]

We have not used the explicit structure of \( g(E) \)!
**HISTORICAL REMARKS**

**K-POINT CORRELATION FUNCTIONS**

- Mehta, Gaudin, Dyson: Gaussian and circular ensembles (60’s and 70’s)
- Mehta and Mahoux: for arbitrary orthogonal and skew orthogonal polynomials (early 90’s)

**Do we really need the Dyson–Mehta–Mahoux integration theorem?**

- Basor and Forrester: CUE by algebraic rearrangement (1994)
  Our method considerably extends this idea and makes the connection to supersymmetry!
Guhr: GUE with supersymmetry (90’s)


Bergere, Ercolani, McLaughlin: biorthogonal polynomial (2000’s)

Borodin, Strahov, Sinclair: with help of random point processes (2000’s)

Akemann, Damgaard, Phillips, Sommers et al.: chiral ensembles in one and two matrix models

Akemann, Forrester, Nagao, Vernizzi et al.: Ginibre ensemble

etc.
Our aim

Example for determinantal structures (schematic)

\[ Z(\kappa) \sim \frac{1}{\sqrt{\text{Ber}_{k/k}(\kappa) \det \left[ \frac{Z(\kappa_a^2, \kappa_b^1)}{\kappa_a^2 - \kappa_b^1} \right]}} \]

Generating functions for \( k \)-point correlations can be expressed by generating functions corresponding to one-point correlation functions!

For Pfaffian structures, we find simplifications similar to this formula!
Two important questions

Have all determinantal and Pfaffian structures the same origin?

If yes:

What are the conditions to find such structures?
Remainder

We consider

\[ Z_N(\kappa, \lambda) \sim \int P(H) \frac{\prod_{n=1}^{k_2} \det(H - \kappa_{n_2}) \prod_{m=1}^{l_2} \det(H^\dagger - \lambda_{m_2})}{\prod_{n=1}^{k_1} \det(H - \kappa_{n_1}) \prod_{m=1}^{l_1} \det(H^\dagger - \lambda_{m_1})} d[H] \]

Reason of determinantal and Pfaffian structures

The integrand already contains a determinantal structure including the characteristic polynomials!
First step: Algebraic rearrangement

Joint probability densities with squared Vandermonde determinant

Multiplying

\[(I) = \frac{\prod_{n=1}^{N} (E_a - \kappa_{n2}) \prod_{m=1}^{l_2} (E^*_a - \lambda_{m2})}{\prod_{a=1}^{k_1} (E_a - \kappa_{n1}) \prod_{m=1}^{l_1} (E^*_a - \lambda_{m1})} |\Delta_N(E)|^2\]

by

\[(II) = \sqrt{\text{Ber}_{K_1/K_2}(\kappa)} \sqrt{\text{Ber}_{L_1/L_2}(\lambda)}\]

yields

\[(I)(II) = \sqrt{\text{Ber}_{K_1/K_2+\mathcal{N}(\kappa, E)} \sqrt{\text{Ber}_{L_1/L_2+\mathcal{N}(\lambda, E^*)}}}\]
SECOND STEP: DETERMINANTAL STRUCTURE

**HERMITIAN SUPERMATRICES**

- symmetric supermatrices with respect to the supergroup $U(p/q)$
  \[ \sigma \in \begin{bmatrix} \text{Herm} (p) & [pqG.v.]^\dagger \\ pqG.v. & \text{Herm} (q) \end{bmatrix} \]

- diagonalization:
  \[ \sigma = \begin{bmatrix} \sigma_1 & \sigma_\eta^\dagger \\ \sigma_\eta & \sigma_2 \end{bmatrix} \rightarrow s = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \]

⇒ measure for the eigenvalues:
\[ d[\sigma] \rightarrow \text{Ber}_{p/q}(s)d[s] \]
\[
\sqrt{\text{Ber}_{p/q}(s)} = \frac{\prod_{1 \leq a < b \leq p} (s_{a1} - s_{b1}) \prod_{1 \leq a < b \leq q} (s_{a2} - s_{b2})}{\prod_{a=1}^{p} \prod_{b=1}^{q} (s_{a1} - s_{b2})} = \frac{\Delta_p(s_1) \Delta_q(s_2)}{\prod_{a=1}^{p} \prod_{b=1}^{q} (s_{a1} - s_{b2})}
\]
SECOND STEP: DETERMINANTAL STRUCTURE

\[ p \leq q \]

**1**

\[
\sqrt{\text{Ber}_{p/q}(s)} \sim \det \begin{bmatrix}
1 \\
\frac{s_{a1} - s_{b2}}{s_{a1} - s_{b2}} \\
\frac{s_{a1} - s_{b2}}{s_{b2}}
\end{bmatrix}
\]

mixes “Cauchy–terms” with “Vandermonde–terms”

**2**

\[
\sqrt{\text{Ber}_{p/q}(s)} \sim \det \begin{bmatrix}
\left(\frac{s_{a1} - \iota \varepsilon}{s_{b2} - \iota \varepsilon}\right)^{p-q} \\
\left(\frac{s_{a1} - \iota \varepsilon}{s_{b2} - \iota \varepsilon}\right)^{p-q} \\
\frac{s_{a1} - s_{b2}}{s_{b2}}
\end{bmatrix}
\]

true for arbitrary \( \varepsilon \), useful for supersymmetric calculations

(1),(2) Kieburg, Guhr\textsuperscript{09} exhibiting the intimate relation to supersymmetry;

(1) Basor, Forrester\textsuperscript{94} without considering the connection to supersymmetry
**Third step: Integration theorems**

**Generalized Andréief’s (1883)**

\[
\int \det \left[ \frac{r_{ab}}{R_b(z_a)} \right] \det \left[ \frac{s_{ab}}{S_b(z_a)} \right] d[z] \\
\sim \det \left[ \begin{array}{c|c} 0 & s_{ab} \\ \hline r_{ba} & \int R_a(z)S_b(z)dz \end{array} \right]
\]

**Generalized de Bruijn’s (1955); for Pfaffian structures**

\[
\int \det \left[ \begin{array}{c|c|c} A_{ab} & B_a(z_b) & C_a(z_b) \end{array} \right] d[z] \\
\sim \text{Pf} \left[ \begin{array}{c|c} 0 & A_{ba} \\ \hline -A_{ab} & \int [B_a(z)C_b(z) - B_b(z)C_a(z)]dz \end{array} \right]
\]

Kieburg, Guhr (2009)
**Fourth Step: Identification of the Kernels**

**Leibniz Expansion of the Determinant**

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det [A - BD^{-1}C]
\]

\[d = N + k_2 - k_1 = N + l_2 - l_1 \geq 0\]

Identification for the cases 
\[(k_1, k_2, l_1, l_2) \in \{(0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 0, 0), (1, 0, 1, 0)\}\]

\[Z_N(\kappa, \lambda) \sim \left[ \begin{array}{c|c} \frac{Z_{d-1}(\kappa b_2, \lambda a_2)}{Z_d(\kappa a_1, \kappa b_2)} & \frac{Z_d(\lambda b_1, \lambda a_2)}{\lambda b_1 - \lambda a_2} \\ \frac{\kappa a_1 - \kappa b_2}{Z_{d+1}(\kappa a_1, \lambda b_1)} & \end{array} \right] \sqrt{\text{Ber} \frac{k_1}{k_2}(\kappa)} \sqrt{\text{Ber} \frac{l_1}{l_2}(\lambda)}\]

\[Z(\kappa, \lambda)\text{ consists of a determinant or Pfaffian determinant containing one and two characteristic polynomial averages!}\]
structures result from an algebraic construction

the joint probability density has only to factorize and the integrals have to be finite, no other requirements

⇒ applicable to a broad class of ensembles
**General Remarks and Conclusions**

### Some Matrix Ensembles Yielding Determinants

<table>
<thead>
<tr>
<th>Matrix Ensemble</th>
<th>Probability Density $P$ for the Matrices</th>
<th>Matrices in the Characteristic Polynomials</th>
<th>Probability Density $g(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermitian ensemble [57, 31, 62, 32, 34, 35]</td>
<td>$\tilde{P}(\text{tr} H^m, m \in \mathbb{N})$ (H = H^\dagger)</td>
<td>$H$</td>
<td>$P(x)\delta(y)$</td>
</tr>
<tr>
<td>Circular unitary ensemble (unitary group) [37, 63, 14, 13, 38, 20, 64, 21]</td>
<td>$\tilde{P}(\text{tr} U^m, m \in \mathbb{N})$ (U^\dagger U = \mathbb{1}_N)</td>
<td>$U$ and $U^\dagger$</td>
<td>$P(e^{i\varphi})\delta(r - 1)$</td>
</tr>
<tr>
<td>Hermitian chiral (complex Laguerre) ensemble [65, 66, 67, 7]</td>
<td>$\tilde{P}(\text{tr}(AA^\dagger)^m, m \in \mathbb{N})$ (A) is a complex (N \times M) matrix with (N \leq M)</td>
<td>$AA^\dagger$</td>
<td>$P(x)x^{M-N}\Theta(x)\delta(y)$</td>
</tr>
<tr>
<td>Gaussian elliptical ensemble [9, 10, 11, 36]; for (\tau = 1)</td>
<td>$\exp\left[ -\frac{(\tau + 1)}{2} \text{tr} H^\dagger H \right] \times \exp\left[ -\frac{(\tau - 1)}{2} \text{Re} \text{ tr} H^2 \right]$ (H) is a complex matrix; (\tau &gt; 0)</td>
<td>$H$ and $H^\dagger$</td>
<td>$\exp\left[ -r^2 (\sin^2 \varphi + \tau \cos^2 \varphi) \right]$</td>
</tr>
<tr>
<td>Complex Ginibre ensemble</td>
<td>$\exp\left[ -\text{tr} A^\dagger A - \text{tr} B^\dagger B \right]$ (C = iA + \mu B) (D = iA^\dagger + \mu B^\dagger) (A) and (B) are complex (N \times M) matrices with (N \leq M)</td>
<td>$CD$ and $D^\dagger C^\dagger$</td>
<td>$K_{M-N} \left( \frac{1 + \mu^2}{2\mu^2} \right) x^{M-N} \times \exp\left( \frac{1 - \mu^2}{2\mu^2} - r \cos \varphi \right)$</td>
</tr>
</tbody>
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Mario Kieburg, Thomas Guhr

Supersymmetry without Supersymmetry
**General Remarks and Conclusions**

### Some Matrix Ensembles Yielding Pfaffian Structures

<table>
<thead>
<tr>
<th>Matrix Ensemble</th>
<th>Probability Density $P$ for the Matrices</th>
<th>Matrices in the Characteristic Polynomials</th>
<th>Probability Densities $g(z_1, z_2)$ and $\tilde{g}(z_1, z_2)$</th>
<th>Probability Density $h(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real symmetric matrices [31, 24, 18]</td>
<td>$\tilde{P} \left( \text{tr} H^m, m \in \mathbb{N} \right)$ [H = H^T = H^*]</td>
<td>$H$</td>
<td>$P(x_1)P(x_2) \times \delta(y_1)\delta(y_2)\Theta(x_2 - x_1)$</td>
<td>$P(x)\delta(y)$</td>
</tr>
<tr>
<td>Circular orthogonal ensemble [4]</td>
<td>$\tilde{P} \left( \text{tr} U^m, m \in \mathbb{N} \right)$ [U^\dagger U = \mathbb{1}_N \text{ and } U^T = U]</td>
<td>$U$ and $U^\dagger$</td>
<td>$P(e^{x_1^\rho})P(e^{x_2^\rho}) \times \delta(r_1 - 1)\delta(r_2 - 1) \times \Theta(\varphi_2 - \varphi_1)$</td>
<td>$P(e^\rho)\delta(r - 1)$</td>
</tr>
<tr>
<td>Real symmetric chiral (real Laguerre) ensemble [21, 32, 33, 34]</td>
<td>$\tilde{P} \left( \text{tr}(AA^T)^m, m \in \mathbb{N} \right)$ [A \text{ is a real } N \times M \text{ matrix with } \nu = M - N \geq 0]</td>
<td>$AA^T$</td>
<td>$P(x_1)P(x_2) \times (x_1x_2)^{(\nu - 1)/2} \times \delta(y_1)\delta(y_2)\Theta(x_2 - x_1)$</td>
<td>$P(x)\delta(y)x^{(\nu - 1)/2}$</td>
</tr>
<tr>
<td>Gaussian real elliptical ensemble; for $\tau = 1$</td>
<td>$\exp \left[ -\frac{(\tau + 1)}{2} \text{tr} H^T H \right] \times \exp \left[ -\frac{(\tau - 1)}{2} \text{tr} H^2 \right]$ [H = H^*; \tau &gt; 0]</td>
<td>$H$</td>
<td>$\prod_{j = 1,2} \exp \left[ -\tau x_j^2 \right] \times \exp \left[ -\tau x_j^2 \right] \delta(y)$</td>
<td>$\exp(\tau x^2)\delta(y)$</td>
</tr>
</tbody>
</table>
| Real Ginibre ensemble [10, 15, 35, 16, 36, 25] [37, 38, 23] | $\exp \left[ -\text{tr} A^T A - \text{tr} B^T B \right]$ \[C = A + \mu B \]
\[D = -A^T + \mu B^T\]
$A$ and $B$ are real $N \times M$ matrices with $\nu = M - N \geq 0$ | $CD$ | $\prod_{j = 1,2} \exp \left[ -2\eta \right] \times \exp \left[ -2\eta \times \delta(y_1)\delta(y_2)\Theta(x_2 - x_1) + 2\eta\delta^2(z_1 - z_2)\Theta(y_1)\right]$ | $x^{\nu/2}\exp \left[ -2\eta \times \right] \times K_{\nu/2}(2\eta\times)\delta(y)$ |
GENERAL REMARKS AND CONCLUSIONS

FOR PFaffIAN AND DETERMINANTAL STRUCTURES

- for many characteristic functions in the denominator:
  the determinantal kernels are Cauchy-terms “$1/(\kappa_a - \kappa_b)$”, moments “$\kappa^{b-1}_a$” and the mean values $Z_1(\kappa_{b1})$ and $Z_2(\kappa_{a1}, \kappa_{b2})$

- a new point of view on the relation between the orthogonal polynomial method and the supersymmetry method

FOR PFaffIAN STRUCTURES

Real and quaternionic ensembles are not distinguishable due to same origin of their Pfaffian structure!
**Other Applications**

- Ensembles in the presence of an external field and intermediate ensembles
- All Efetov–Wegner terms for the supersymmetric Itzykson–Zuber integral
- Supersymmetry calculations in general
GENERAL REMARKS AND CONCLUSIONS

OTHER APPLICATIONS

We use structures of supersymmetry without ever mapping our integrals onto superspace!

⇒ Supersymmetry without supersymmetry
THANK YOU FOR YOUR ATTENTION!