"Supersymmetry without Supersymmetry": A new approach to derive determinantal and Pfaffian structures

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Average over rotation invariant matrix ensembles are interesting for:

- disordered systems
- quantum chaos
- number theory
- matrix models in high energy physics
- Weyl's character formula
- etc.

SETTING

$$Z(\kappa,\lambda) \sim \int P(H) \frac{\prod\limits_{n=1}^{k_2}\det(H-\kappa_{n2})\prod\limits_{m=1}^{l_2}\det(H^\dagger-\lambda_{m2})}{\prod\limits_{n=1}^{k_1}\det(H-\kappa_{n1})\prod\limits_{m=1}^{l_1}\det(H^\dagger-\lambda_{m1})} d[H]$$

+ factorizable probability density P

$$P(E)=\prod_{j=1}^{N}P(E_j) \quad , \text{ with } E=\mathrm{diag}\,(E_1,\ldots,E_N)$$

H has not to be symmetric!

$$Z_N(\kappa) = \int\limits_{\text{Herm}(N)} P(H) \prod_{j=1}^{k} \frac{\det(H-\kappa_{j2})}{\det(H-\kappa_{j1})} d[H]$$

$$\sim \int \prod_{a=1}^{N} \left[P(E_a) \prod_{j=1}^{k} \frac{E_a-\kappa_{j2}}{E_a-\kappa_{j1}} \right] |\Delta_N(E)|^2 d[E]$$

It is well known that this generating function exhibits a determinantal structure!
Baik, Deift, Strahov (2003); Grönqvist, Guhr, Kohler (2004); Borodin, Strahov (2005); Guhr (2006)

ALGEBRAIC REARRANGEMENT

Multiplying

$$(I) = \prod_{a=1}^{N} \prod_{j=1}^{k} \frac{E_a - \kappa_{j2}}{E_a - \kappa_{j1}} \Delta_N(E)$$

by

$$(II) = \sqrt{\mathrm{Ber}_{k/k}(\kappa)} \sim \det\left[\frac{1}{\kappa_{a1} - \kappa_{b2}}\right]$$

yields

$$(I)(II) = \sqrt{\mathrm{Ber}_{k/k+N}(\kappa, E)} \sim \det\left[\begin{array}{c|c} \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & \dfrac{1}{\kappa_{a1} - E_b} \\ \hline {\kappa_{b2}}^{a-1} & {E_b}^{a-1} \end{array}\right]$$

The variables κ_{a2} and E_b are the new fermionic eigenvalues!

$$Z_N(\kappa) \sim \int \prod_{a=1}^{N} P(E_a) \frac{\sqrt{\mathrm{Ber}_{k/k+N}(\kappa, E)}}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \Delta_N(E) d[E]$$

DETERMINANTAL STRUCTURE

$$Z_N(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}}$$

$$\times \det \left[\begin{array}{c|c} \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & \displaystyle\int \frac{P(E)E^{b-1}}{\kappa_{a1} - E} dE \\ \hline \kappa_{b2}{}^{a-1} & \int P(E)E^{a+b-2} dE \end{array} \right]$$

LEIBNIZ EXPANSION OF THE DETERMINANT

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \ \det[A - BD^{-1}C]$$

yields a $k \times k$ determinant

COMPARING THE DETERMINANT KERNEL FOR $k = 1$

$$Z_N(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \det \left[\frac{Z_N(\kappa_{a1}, \kappa_{b2})}{\kappa_{a1} - \kappa_{b2}} \right]$$

We have not used the explicit structure of $P(E)$!
$\rightarrow$ These structures have to be true for other matrix ensembles!

SECOND EXAMPLE: SPECIAL ORTHOGONAL GROUP SO $(2N)$

$$Z_{2N}(\kappa) = \int\limits_{SO(2N)} \prod_{j=1}^{k} \frac{\det(O-\kappa_{j2})}{\det(O-\kappa_{j1})} d\mu(O)$$

$$\sim \operatorname{Sdet}^{-N}\kappa \int \prod_{a=1}^{N} \left[g(E_a) \prod_{j=1}^{k} \frac{E_a - \tilde{\kappa}_{j2}}{E_a - \tilde{\kappa}_{j1}} \right] |\Delta_N(E)|^2 d[E]$$

with

$$g(E_a) \sim \frac{\Theta(E_a - 1)\Theta(1 - E_a)}{\sqrt{1 - E_a{}^2}} \quad ; \quad \tilde{\kappa}_{ji} = \frac{\kappa_{ji} + \kappa_{ji}^{-1}}{2}$$

Interesting for mathematicians: Weyl–type character formula

Huckleberry, Püttmann, Zirnbauer (2005)

Algebraic rearrangement

Multiplying

$$(I) = \prod_{a=1}^{N} \prod_{j=1}^{k} \frac{E_a - \tilde{\kappa}_{j2}}{E_a - \tilde{\kappa}_{j1}} \Delta_N(E)$$

by

$$(II) = \sqrt{\mathrm{Ber}_{k/k}(\tilde{\kappa})}$$

yields

$$(I)(II) = \sqrt{\mathrm{Ber}_{k/k+N}(\tilde{\kappa}, E)} \sim \det \left[\begin{array}{c|c} \dfrac{1}{\tilde{\kappa}_{a1} - \tilde{\kappa}_{b2}} & \dfrac{1}{\tilde{\kappa}_{a1} - E_b} \\ \hline {\tilde{\kappa}_{b2}}^{a-1} & {E_b}^{a-1} \end{array} \right]$$

The variables $\tilde{\kappa}_{a2}$ and E_b are the new fermionic eigenvalues!

Second example: Special orthogonal group SO $(2N)$

$$Z_{2N}(\kappa) \sim \operatorname{Sdet}^{-N}\kappa \int \prod_{a=1}^{N} g(E_a) \frac{\sqrt{\operatorname{Ber}_{k/k+N}(\tilde{\kappa}, E)}}{\sqrt{\operatorname{Ber}_{k/k}(\tilde{\kappa})}} \Delta_N(E) d[E]$$

Determinantal structure

$$Z_{2N}(\kappa) \sim \frac{\operatorname{Sdet}^{-N}\kappa}{\sqrt{\operatorname{Ber}_{k/k}(\tilde{\kappa})}}$$

$$\times \quad \det \left[\begin{array}{c|c} \dfrac{1}{\tilde{\kappa}_{a1} - \tilde{\kappa}_{b2}} & \displaystyle\int \frac{g(E)E^{b-1}}{\tilde{\kappa}_{a1} - E} dE \\ \hline \tilde{\kappa}_{b2}^{a-1} & \displaystyle\int g(E) E^{a+b-2} dE \end{array} \right]$$

Second example: Special orthogonal group SO $(2N)$

Leibniz expansion of the determinant

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \ \det[A - BD^{-1}C]$$

Comparing the determinant kernel for $k = 1$

$$Z_{2N}(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\tilde{\kappa})}} \det \left[\frac{Z_{2N}(\kappa_{a1}, \kappa_{b2})}{\tilde{\kappa}_{a1} - \tilde{\kappa}_{b2}} \right]$$

We have not used the explicit structure of $g(E)$!

k-point correlation functions

- Mehta, Gaudin, Dyson: Gaussian and circular ensembles (60's and 70's)
- Mehta and Mahoux: for arbitrary orthogonal and skew orthogonal polynomials (early 90's)

Do we really need the Dyson–Mehta–Mahoux integration theorem?

- Basor and Forrester: CUE by algebraic rearrangement (1994)
  Our method considerably extends this idea and makes the connection to supersymmetry!

Historical remarks

- Guhr: GUE with supersymmetry (90's)
- Baik, Deift, and Strahov: characteristic polynomials of random Hermitian matrices (2003)
- Guhr: arbitrary unitary rotation invariant ensembles with factorizing superfunction (2006)
- Bergere, Ercolani, McLaughlin: biorthogonal polynomial (2000's)
- Borodin, Strahov, Sinclair: with help of random point processes (2000's)
- Akemann, Damgaard, Phillips, Sommers et al.: chiral ensembles in one and two matrix models
- Akemann, Forrester, Nagao, Vernizzi et al.: Ginibre ensemble
- etc.

Our aim

Example for determinantal structures (schematic)

$$Z(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \det\left[\frac{Z(\kappa_{a2}, \kappa_{b1})}{\kappa_{a2} - \kappa_{b1}}\right]$$

Generating functions for k-point coorelations can be expressed by generating functions corresponding to one-point correlation functions!

For Pfaffian structures, we find simplifications similar to this formula!

Have all determinantal and Pfaffian structures the same origin?

IF YES:

What are the conditions to find such structures?

Reminder

We consider

$$Z_N(\kappa,\lambda) \sim \int P(H)\frac{\prod\limits_{n=1}^{k_2}\det(H-\kappa_{n2})\prod\limits_{m=1}^{l_2}\det(H^\dagger-\lambda_{m2})}{\prod\limits_{n=1}^{k_1}\det(H-\kappa_{n1})\prod\limits_{m=1}^{l_1}\det(H^\dagger-\lambda_{m1})}d[H]$$

Reason of determinantal and Pfaffian structures

The integrand already contains a determinantal structure including the characteristic polynomials!

JOINT PROBABILITY DENITIES WITH SQUARED VANDERMONDE DETERMINANT

Multiplying

$$(I) = \prod_{a=1}^{N} \frac{\prod_{n=1}^{k_2}(E_a - \kappa_{n2}) \prod_{m=1}^{l_2}(E_a^* - \lambda_{m2})}{\prod_{n=1}^{k_1}(E_a - \kappa_{n1}) \prod_{m=1}^{l_1}(E_a^* - \lambda_{m1})} |\Delta_N(E)|^2$$

by

$$(II) = \sqrt{\mathrm{Ber}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}_{l_1/l_2}(\lambda)}$$

yields

$$(I)(II) = \sqrt{\mathrm{Ber}_{k_1/k_2+N}(\kappa, E)}\sqrt{\mathrm{Ber}_{l_1/l_2+N}(\lambda, E^*)}$$

HERMITIAN SUPERMATRICES

- symmetric supermatrices with respect to the supergroup $U(p/q)$

$$\sigma \in \left[\begin{array}{c|c} \mathrm{Herm}\,(p) & [pq\mathrm{G.v.}]^\dagger \\ \hline pq\mathrm{G.v.} & \mathrm{Herm}\,(q) \end{array}\right]$$

- diagonalization:

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_\eta^\dagger \\ \sigma_\eta & \sigma_2 \end{bmatrix} \longrightarrow s = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

$\Rightarrow$ measure for the eigenvalues:

$$d[\sigma] \longrightarrow \mathrm{Ber}_{p/q}(s) d[s]$$

Explicit expression

$$\sqrt{\mathrm{Ber}_{p/q}(s)} = \frac{\prod\limits_{1\leq a<b\leq p}(s_{a1}-s_{b1})\prod\limits_{1\leq a<b\leq q}(s_{a2}-s_{b2})}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(s_{a1}-s_{b2})}$$

$$= \frac{\Delta_p(s_1)\Delta_q(s_2)}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(s_{a1}-s_{b2})}$$

$p \leq q$

(1)

$$\sqrt{\mathrm{Ber}_{p/q}(s)} \sim \det \left[\begin{array}{c} \dfrac{1}{s_{a1} - s_{b2}} \\ s_{b2}^{a-1} \end{array} \right]$$

mixes "Cauchy–terms" with "Vandermonde–terms"

(2)

$$\sqrt{\mathrm{Ber}_{p/q}(s)} \sim \det \left[\begin{array}{c} \left(\dfrac{s_{a1} - \imath\varepsilon}{s_{b2} - \imath\varepsilon}\right)^{p-q} \dfrac{1}{s_{a1} - s_{b2}} \\ s_{b2}^{a-1} \end{array} \right]$$

true for arbitrary ε, useful for supersymmetric calculations

(1),(2) Kieburg, Guhr[09] exhibiting the intimate relation to supersymmetry;

(1) Basor, Forrester[94] without considering the connection to supersymmetry

Third step: integration theorems

Generalized Andréief's (1883)

$$\int \det\left[\frac{r_{ab}}{R_b(z_a)}\right] \det\left[\frac{s_{ab}}{S_b(z_a)}\right] d[z]$$

$$\sim \quad \det\left[\begin{array}{c|c} 0 & s_{ab} \\ \hline r_{ba} & \int R_a(z)S_b(z)dz \end{array}\right]$$

Generalized de Bruijn's (1955); for Pfaffian structures

$$\int \det\left[\begin{array}{c|c|c} A_{ab} & B_a(z_b) & C_a(z_b) \end{array}\right] d[z]$$

$$\sim \quad \mathrm{Pf}\left[\begin{array}{c|c} 0 & A_{ba} \\ \hline -A_{ab} & \int \left[B_a(z)C_b(z) - B_b(z)C_a(z)\right] dz \end{array}\right]$$

Kieburg, Guhr (2009)

Fourth step: identification of the kernels

Leibniz expansion of the determinant

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \ \det[A - BD^{-1}C]$$

$d = N + k_2 - k_1 = N + l_2 - l_1 \geq 0$

Identification for the cases
$(k_1, k_2, l_1, l_2) \in \{(0,1,0,1),(0,0,1,1),(1,1,0,0),(1,0,1,0)\}$

$$Z_N(\kappa,\lambda) \sim \frac{\det\left[\begin{array}{c|c} Z_{d-1}(\kappa_{b2},\lambda_{a2}) & \frac{Z_d(\lambda_{b1},\lambda_{a2})}{\lambda_{b1}-\lambda_{a2}} \\ \hline \frac{Z_d(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}} & Z_{d+1}(\kappa_{a1},\lambda_{b1}) \end{array}\right]}{\sqrt{\mathrm{Ber}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}_{l_1/l_2}(\lambda)}}$$

$Z(\kappa,\lambda)$ **consists of a determinant or Pfaffian determinant containing one and two characteristic polynomial averages!**

FOR PFAFFIAN AND DETERMINANTAL STRUCTURES

- structures result from an algebraic construction
- the joint probability density has only to factorize and the integrals have to be finite, **no other requirements**

$\Rightarrow$ applicable to a broad class of ensembles

SOME MATRIX ENSEMBLES YIELDING DETERMINANTS

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $g(z)$
Hermitian ensemble [57, 31, 62, 32, 34, 35]	$\widetilde{P}(\operatorname{tr} H^m, m \in \mathbb{N})$ $H = H^\dagger$	H	$P(x)\delta(y)$
circular unitary ensemble (unitary group) [37, 63, 14, 13, 38, 20, 64, 21]	$\widetilde{P}(\operatorname{tr} U^m, m \in \mathbb{N})$ $U^\dagger U = \mathbb{1}_N$	U and $U^\dagger$	$P(e^{\imath\varphi})\,\delta(r-1)$
Hermitian chiral (complex Laguerre) ensemble [65, 66, 67, 7]	$\widetilde{P}(\operatorname{tr}(AA^\dagger)^m, m \in \mathbb{N})$ A is a complex $N \times M$ matrix with $N \leq M$	$AA^\dagger$	$P(x)x^{M-N}\Theta(x)\delta(y)$
Gaussian elliptical ensemble [9, 10, 11, 36]; for $\tau = 1$ complex Ginibre ensemble	$\exp\left[-\frac{(\tau+1)}{2}\operatorname{tr} H^\dagger H\right] \times$ $\times \exp\left[-\frac{(\tau-1)}{2}\operatorname{Re}\ \operatorname{tr} H^2\right]$ H is a complex matrix; $\tau > 0$	H and $H^\dagger$	$\exp\left[-r^2\left(\sin^2\varphi + \tau\cos^2\varphi\right)\right]$
Gaussian complex chiral ensemble [12]	$\exp\left[-\operatorname{tr} A^\dagger A - \operatorname{tr} B^\dagger B\right]$ $C = \imath A + \mu B$ $D = \imath A^\dagger + \mu B^\dagger$ A and B are complex $N \times M$ matrices with $N \leq M$	CD and $D^\dagger C^\dagger$	$K_{M-N}\left(\frac{1+\mu^2}{2\mu^2}r\right)r^{M-N}\times$ $\times \exp\left(\frac{1-\mu^2}{2\mu^2}r\cos\varphi\right)$

General remarks and conclusions

Some matrix ensembles yielding Pfaffian structures

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability densities $g(z_1,z_2)$ and $\bar{g}(z_1,z_2)$	probability density $h(z)$
real symmetric matrices [31, 24, 18]	$\widetilde{P}\,(\mathrm{tr}\,H^m, m\in\mathbb{N})$ $H=H^T=H^*$	H	$P(x_1)P(x_2)\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2-x_1)$	$P(x)\delta(y)$
circular orthogonal ensemble [4]	$\widetilde{P}\,(\mathrm{tr}\,U^m, m\in\mathbb{N})$ $U^\dagger U=\mathbb{1}_N$ and $U^T=U$	U and $U^\dagger$	$P(e^{\imath\varphi_1})P(e^{\imath\varphi_2})\times$ $\times\delta(r_1-1)\delta(r_2-1)\times$ $\times\Theta(\varphi_2-\varphi_1)$	$P(e^{\imath\varphi})\delta(r-1)$
real symmetric chiral (real Laguerre) ensemble [21, 32, 33, 34]	$\widetilde{P}\,\left(\mathrm{tr}(AA^T)^m, m\in\mathbb{N}\right)$ A is a real $N\times M$ matrix with $\nu=M-N\geq 0$	AA^T	$P(x_1)P(x_2)\times$ $\times(x_1x_2)^{(\nu-1)/2}\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2-x_1)$	$P(x)\delta(y)x^{(\nu-1)/2}$
Gaussian real elliptical ensemble; for $\tau=1$ real Ginibre ensemble [10, 15, 35, 16, 36, 25] [37, 38, 23]	$\exp\left[-\frac{(\tau+1)}{2}\,\mathrm{tr}\,H^TH\right]\times$ $\times\exp\left[-\frac{(\tau-1)}{2}\,\mathrm{tr}\,H^2\right]$ $H=H^*$; $\tau>0$	H	$\prod\limits_{j\in\{1,2\}}\exp\left[-\tau x_j^2\right]\times$ $\times\sqrt{\mathrm{erfc}(\sqrt{2(1+\tau)}y_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2-x_1)+$ $+2\imath\delta^2(z_1-z_2^*)\Theta(y_1)]$	$\exp(-\tau x^2)\delta(y)$
Gaussian real chiral ensemble [9, 39]	$\exp\left[-\,\mathrm{tr}\,A^TA-\mathrm{tr}\,B^TB\right]$ $C=A+\mu B$ $D=-A^T+\mu B^T$ A and B are real $N\times M$ matrices with $\nu=M-N\geq 0$	CD	$\prod\limits_{j\in\{1,2\}}\exp\left[-2\eta_-z_j\right]\times$ $\times\lvert z_j\rvert^\nu\sqrt{f(2\eta_+z_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2-x_1)+$ $+2\imath\delta^2(z_1-z_2^*)\Theta(y_1)]$	$x^{\nu/2}\exp\left[-2\eta_-x\right]\times$ $\times K_{\nu/2}(2\eta_+x)\delta(y)$

General remarks and conclusions

For Pfaffian and determinantal structures

- **for many characteristic functions in the denominator:** the determinantal kernels are Cauchy-terms "$1/(\kappa_{a1} - \kappa_{b2})$", moments "$\kappa_{a1}^{b-1}$" and the mean values $Z_1(\kappa_{b1})$ and $Z_2(\kappa_{a1}, \kappa_{b2})$
- a new point of view on the relation between the orthogonal polynomial method and the supersymmetry method

For Pfaffian structures

Real and quaternionic ensembles are not distinguishable due to same origin of their Pfaffian structure!

OTHER APPLICATIONS

- ensembles in the presence of an external field and intermediate ensembles
- all Efetov–Wegner terms for the supersymmetric Itzykson–Zuber integral
- supersymmetry calculations in general

Other applications

We use structures of supersymmetry without ever mapping our integrals onto superspace!

$\Rightarrow$**Supersymmetry without supersymmetry**

THANK YOU FOR YOUR ATTENTION!

- M. Kieburg and T. Guhr. "Derivation of determinantal structures for random matrix ensembles in a new way" submitted to *J. Phys.* **A** (2009)
  preprint: arXiv:0912.0654
- M. Kieburg and T. Guhr. "A new approach to derive Pfaffian structures for random matrix ensembles" submitted to *J. Phys.* **A** (2009)
  preprint: arXiv:0912.0658