ARBITRARY ROTATION INVARIANT RANDOM MATRIX ENSEMBLES AND SUPERSYMMETRY

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Random Matrix Theory plays an important role in:

- disordered systems
- quantum chaos
- number theory
- analysis of correlations in networks, etc.

Object of interest: probability density $P(H)$ over a matrix set

Introduction

- Efetov's approach for Gaussian matrix ensembles (early 80's)

Mathematical question: Is the supersymmetry method restricted to Gaussian matrix ensemles?

- proof of the universality on the local mean level spacing by Hackenbroich and Weidenmueller (1995)
- first approaches of superbosonization by
  - Lehmann, Saher, Sokolov, and Sommers: chaotic scattering (1995)
  - Efetov, Schwiete, and Takahashi: disordered and chaotic systems (2004)

Introduction

Generalized Hubbard–Stratonovich transformation

arbitrary rotation invariant ensembles on symmetric matrices

- unitary symmetry: Guhr (2006); partly heuristic
- orthogonal and unitary symplectic symmetry: Kieburg, Grönqvist, and Guhr (2008); mathematically more rigorous

Superbosonization Formula

arbitrary rotation invariant ensembles on symmetric matrices

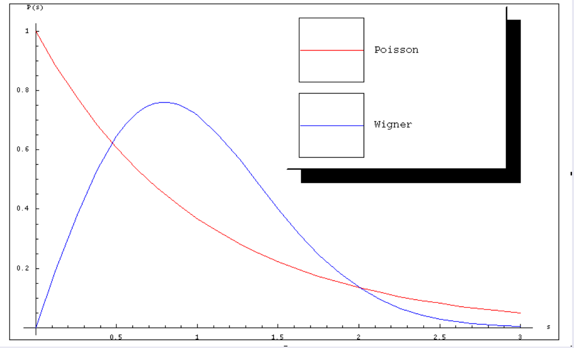
- orthogonal symmetry: Sommers (2007); idea of the proof
- of all symmetry classes: Littelmann, Sommers, and Zirnbauer (2007/2008); mathematically rigorous

Comparison of both approaches

Kieburg, Sommers, and Guhr (2009); for all symmetry classes

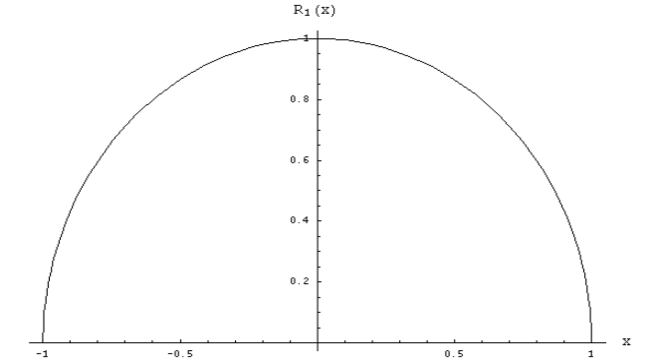
What do we want to measure and calculate:

- k-point level correlation function
- level spacing
- moments of characteristic polynomials etc.



What is well analysed:

- factorizable probability densities $P(H) = \prod_{l=1}^{N} \tilde{P}(E_l)$
- in particular: Gaussian distributions $P(H) \sim e^{-Tr(H^2)}$



arbitrary invariant random matrix ensembles are interesting for:

- financial correlation matrices
- high-energy physics and quantum gravity
- large-scale behavior and universality

The goal:

- a supersymmetric representation of

$$Z(x^-) \sim \int P(H) \frac{\prod_{n=1}^{q} \det(H - x_n^-)^{\lambda_2}}{\prod_{m=1}^{p} \det(H - x_m^-)^{\lambda_1}} d[H]$$

λ_j depends on the Dyson–index $\beta \in \{1, 2, 4\}$

The generalized Hubbard–Stratonovich transformation

Two Fourier–transformations in superspace

$$\int \Phi(B)d[z,\eta] \sim \int \frac{\det \rho_1^{\kappa}\delta(r_2)}{|\Delta(r_2)|^{4/\beta}} D_{r_2}\Phi\left(\begin{bmatrix}\rho_1 & \rho_\eta \\ -\rho_\eta^\dagger & \rho_2 - \rho_\eta^\dagger \rho_1^{-1}\rho_\eta\end{bmatrix}\right) d[\rho]$$

D_{r_2} is a power of a differential operator analogous to the Sekiguchi–differential operator

$$\frac{1}{\Delta(r_2)} \det\left[r_{n2}^{q-m}\left(\frac{\partial}{\partial r_{n2}} + (q-m)\frac{2}{\beta}\frac{1}{r_{n2}}\right)\right]$$

Remarks:

- exponent κ is function of N, p, q and β with $p \geq N$
- Fermion–Fermion block ρ_2 runs in a non-compact domain

$$\mathrm{Herm}\,(1,N) \rightarrow \left[\begin{array}{c|cc} \mathrm{Herm}\,(1,p) & pq\ G.v. & [pq\ G.v.]^* \\ \hline -[pq\ G.v.]^\dagger & \multicolumn{2}{c}{\multirow{2}{*}{\mathrm{Herm}\,(4,q)}} \\ [pq\ G.v.]^T & & \end{array}\right]$$

$$\mathrm{Herm}\,(2,N) \rightarrow \left[\begin{array}{c|c} \mathrm{Herm}\,(2,p) & pq\ G.v. \\ \hline -[pq\ G.v.]^\dagger & \mathrm{Herm}\,(2,q) \end{array}\right]$$

$$\mathrm{Herm}\,(4,N) \rightarrow \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{Herm}\,(4,p)} & \begin{array}{c} pq\ G.v. \\ [pq\ G.v.]^* \end{array} \\ \hline -[pq\ G.v.]^\dagger & [pq\ G.v.]^T & \mathrm{Herm}\,(1,q) \end{array}\right]$$

$\mathrm{Herm}\,(\beta,.)$: set of real symmetric ($\beta=1$), hermitian ($\beta=2$) and quaternionic selfadjoint ($\beta=4$) matrices

Guhr[06]; Kieburg, Grönqvist, Guhr[08]; Kieburg, Sommers, Guhr[09]

Two Fourier–transformations in superspace

$$\int \Phi(B) d[z,\eta] \sim \int \mathrm{Sdet}\, \rho^{\kappa} \Phi(\rho)\, d[\rho]$$

Remarks:

- same exponent κ as for the generalized Hubbard–Stratonovich transformation
- Fermion–Fermion block ρ_2 runs in a compact domain

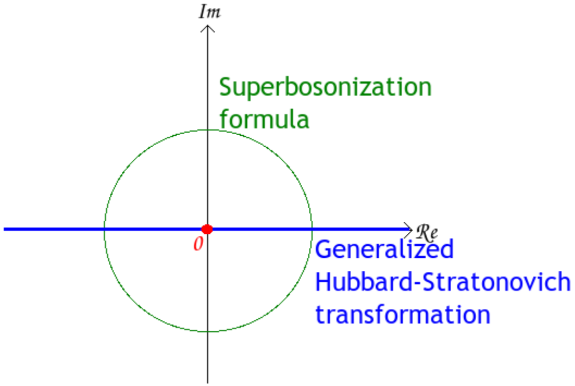
$$\mathrm{Herm}(1,N) \rightarrow \left[\begin{array}{c|cc} \mathrm{Herm}(1,p) & pq\ G.v. & [pq\ G.v.]^* \\ \hline -[pq\ G.v.]^\dagger & \multicolumn{2}{c}{\multirow{2}{*}{\mathrm{CU}(4,q)}} \\ [pq\ G.v.]^T & & \end{array}\right]$$

$$\mathrm{Herm}(2,N) \rightarrow \left[\begin{array}{c|c} \mathrm{Herm}(2,p) & pq\ G.v. \\ \hline -[pq\ G.v.]^\dagger & \mathrm{CU}(2,q) \end{array}\right]$$

$$\mathrm{Herm}(4,N) \rightarrow \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{Herm}(4,p)} & \begin{array}{c} pq\ G.v. \\ [pq\ G.v.]^* \end{array} \\ \hline -[pq\ G.v.]^\dagger & [pq\ G.v.]^T & \mathrm{CU}(1,q) \end{array}\right]$$

$\mathrm{CU}(\beta,.)$: set of circular orthogonal (COE, $\beta = 1$), unitary (CUE, $\beta = 2$) and unitary-symplectic (CSE, $\beta = 4$) ensembles

Sommers[07]; Bunder et al.[07]; Littelmann, Sommers, Zirnbauer[08]; Kieburg, Sommers, Guhr[09]



Eigenvalue integrals in the Fermion–Fermion block: the superbosonization formula is the contour integral of the generalized Hubbard–Stratonovich transformation

Basile, Akemann[07]; Kieburg, Sommers, Guhr[09]

Thank you for your attention!

- M. Kieburg, J. Grönqvist and T. Guhr. “Arbitrary rotation invariant random matrix ensembles and supersymmetry: orthogonal and unitary–symplectic case” submitted to *J. Phys.* **A** (2008)
- M. Kieburg, H.-J. Sommers and T. Guhr. “Comparison of the superbosonization formula and the generalized Hubbard–Stratonovich transformation” submitted to *J. Phys.* **A** (2009)