Cartesian integration of Grassmann variables over invariant functions

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- Cauchy–like integral theorems by Efetov, Wegner, Constantinescu, de Groote (80's)
- connection between change of coordinates in superspace and differential operators comprised in the Berezin–measure Rothstein

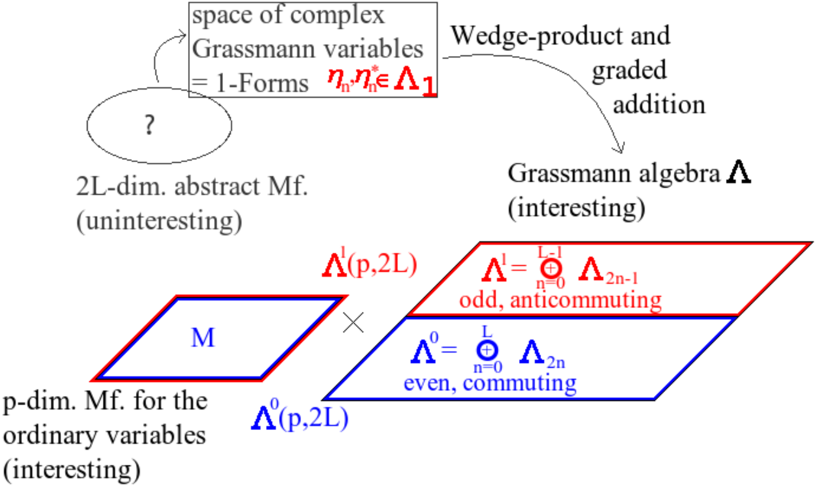
Supersymmetry plays an important role in:

- field theory
- random matrix theory
- mesoscopic physics etc.

Mathematical objects:

- commuting (ordinary) variables: $[x_n, x_m]_- = 0$
- anticommuting (Grassmann) variables: $[\eta_n, \eta_m]_+ = 0$

RIEMANNIAN SUPERSPACE



Considering the superspace

$$\mathfrak{L}(p,2L) = \left(\Lambda^0(p,2L)\right)^p \times \left(\Lambda^1(p,2L)\right)^{2L}$$

Let M be a vectorspace.

A superfunction f on $\mathfrak{L}(p,2L)$ can be expanded in a power series

$$f(x,\eta)=\sum_{j_1,j_2} f_{j_1,j_2}(x)\prod_{n=1}^{L}(\eta_n^*)^{j_{1n}}\eta_n^{j_{2n}}$$

Diagonal constant metric on $\mathfrak{L}(p,2L)$

$$g\left(a,\alpha;b,\beta\right)=\sum_{n=1}^{p} g_n a_n b_n+\sum_{m=1}^{L} h_m\left(\alpha_m^*\beta_m+\beta_m^*\alpha_m\right)$$

$$g_n\in\mathbb{R}^+\quad\text{and}\quad h_m\in\mathbb{C}$$

- Integration over η is discrete: $\int \eta_n{}^j d\eta_m = \frac{\delta_{1j}\delta_{nm}}{\sqrt{2\pi}}$
- $d\eta_m$ is not a differential as for ordinary variables
- integration = differentiation = interior product

$$\sqrt{2\pi}\int f(\eta)d\eta_n = \frac{\partial}{\partial\eta_n}f(\eta) = \iota_{e_n}f(\eta)$$

$\Rightarrow$ Integration over Grassmann variables is connected with a differential operator

Is there a compact form of this differential operator?

Efetov[83]; Wegner[83]; Rothstein[87]; Kieburg, Kohler, Guhr[08]

INTEGRATION THEOREM

Superfunction invariant under $f(x,\eta) = f(r(x,\eta),0)$ with differentiable mapping $r : \mathfrak{L}(p,2L) \to \left(\Lambda^0(p,2L)\right)^p$

$$\int f(x,\eta)d[\eta] = D_r f(r,0)$$

$$D_r \sim \sum_{n=0}^{L} \binom{L}{n} \Delta_{C,r}^{L-n}(-\Delta_{S,r})^n$$

$$\Delta_C = \sum_{n=1}^{p} \frac{1}{g_n}\frac{\partial^2}{\partial {x_n}^2} \quad \text{and} \quad \Delta_S = \Delta_C + 2\sum_{m=1}^{L}\frac{1}{h_m}\frac{\partial^2}{\partial\eta_m\partial\eta_m^*}$$

- right hand side is independent of Grassmann variables
- D_r is uniquely defined by invariance class
- D_r is a differential operator of order L

Kieburg, Kohler, Guhr[08]

- real supervector $v = (a_1, \ldots, a_p, \alpha_1, \ldots, \alpha_L, \alpha_1^*, \ldots, \alpha_L^*)^T$
- f is $\mathrm{UOSp}^{(+)}(p/2L)$ invariant: $f(v) = f(Uv)$
- f only depends on $r^2 = v^\dagger v$

$\Rightarrow$ very compact form of $D_r \sim \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L$

For $p = 2L$, we find

$$\int f(v)d[v] \sim \int r^{2L-1} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^L f(r)dr \sim f(0)$$

- similar results for rotation invariant superfunctions on complex and quaternionic supervectors

Efetov[83]; Wegner[83]; Constantinescu, de Groote[89]; Kieburg, Kohler, Guhr[08]

- $\mathrm{U}\,(k_1/k_2)$–symmetric supermatrix $\sigma = \begin{bmatrix} \sigma_1 & \eta^\dagger \\ \eta & \sigma_2 \end{bmatrix}$
- f is $\mathrm{U}\,(k_1/k_2)$ invariant: $f(\sigma) = f(U\sigma U^\dagger)$
- f only depends on $\mathrm{Str}\,\sigma^m \Rightarrow$ eigenvalues s of σ

$$D_s \sim \frac{1}{\Delta(s_1)\Delta(s_2)} \times$$
$$\sum_{n=1}^{k_1k_2} \binom{k_1k_2}{n} \left(\mathrm{Str}\,\frac{\partial^2}{\partial s^2}\right)^n \frac{\Delta(s_1)\Delta(s_2)}{\det\left(\frac{1}{s_1-\imath s_2}\right)} \left(-\mathrm{Str}\,\frac{\partial^2}{\partial s^2}\right)^{k_1k_2-n} \det\left(\frac{1}{s_1-\imath s_2}\right)$$

For $k_1 = k_2$, we find

$$\int f(\sigma) d[\sigma] \sim f(0)$$

Results for rotation invariant superfunctions on $\mathrm{UOSp}^{(\pm)}(k_1/k_2)$–symmetric supermatrices are similar. Their differential operators are more complex.

COUNTER–EXAMPLE

$\mathrm{UOSp}^{(+)}(1/2)$–invariant superfunctions on $\sigma = \begin{bmatrix} s_1 & \eta & \eta^* \\ -\eta^* & s_2 & 0 \\ \eta & 0 & s_2 \end{bmatrix}$

- no Cauchy–like theorem

$\Rightarrow$ Rotation invariance and same number of ordinary and Grassmann–variables are no sufficient conditions!

Wegner[83]; Constantinescu, de Groote[89]; Kieburg, Kohler, Guhr[08]

- supermatrix Bessel function for $\mathrm{U}\,(k_1/k_2)$:
  $\Phi(s,r) = \int\limits_{\mathrm{U}\,(k_1/k_2)} \exp \mathrm{Str}\, sUrU^\dagger d\mu(U)$
- implicit definition with rotation invariant f

$$\int f(\sigma) \exp \mathrm{Str}\, \sigma r d[\sigma] = \int f(s)\Phi(s,r) d[s] + \mathrm{b.t.}$$

Procedure

- integration over Grassmann–variables

$\Rightarrow$ differential operator acting only on f

- ordinary group integrals over $\mathrm{U}\,(k_1)$ and $\mathrm{U}\,(k_2)$
- partial integration under the eigenvalue integrals

RESULTS

We find

- the supermatrix Bessel function is equal to a differential operator acting on the product of two ordinary matrix Bessel functions
- Efetov–Wegner terms (boundary terms "b.t.")

DIFFICULTIES

- compact expression for the supermatrix Bessel function of $\mathrm{U}\left(k_1/k_2\right)$

BUT no explicit expressions of the ordinary matrix Bessel functions of $\mathrm{O}(n)$ and $\mathrm{USp}(n)$

$\Rightarrow$ no explicit expressions of the supermatrix Bessel functions of $\mathrm{UOSp}^{(\pm)}(k_1/k_2)$

Kieburg, Kohler, Guhr[08]

Thank you for your attention!

- M. Kieburg, H. Kohler and T. Guhr. “Integration of Grassmann variables over invariant functions on flat superspaces”, *J. Math. Phys.* **50**, 013528 (2008)