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Random Matrix Theory for the Wilson-Dirac operator

Mario Kieburg

Department of Physics and Astronomy
SUNY Stony Brook (NY, USA)

Bielefeld, December 14th, 2011

Outline

- Introduction in Lattice QCD and in the Random Matrix Model
- The Joint Probability Densities
- The Eigenvalue Densities
- Conclusions and Outlook

Introduction

in Lattice QCD
and
in the Random Matrix Model

Action of continuums QCD

The partition function of N_{f} fermionic flavors

$$Z = \int \exp\left[-\imath S_{\mathrm{YM}}(A) - \imath \sum_{j=1}^{N_{\mathrm{f}}} \int \bar{\psi}_j(\imath D(A) - m_j)\psi_j d^4x\right] \mathrm{D}[A, \psi]$$

The Yang-Mills action of $\mathrm{SU}(3)$

$$S_{\mathrm{YM}}(A) = \frac{1}{4g^2} \int \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} d^4x$$

with the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]$$

The four components of the vector potential $A_\mu \in \mathrm{su}(3)$ are 3×3 matrix valued functions.

The continuum Dirac-operator

Fermionic fields ψ_j are Grassmann variables

$$\Rightarrow Z = \int \prod_{j=1}^{N_\mathrm{f}} \det\left(\imath D(A) - m_j\right) \exp\left[-\imath S_{\mathrm{YM}}(A)\right] \mathrm{D}[A]$$

The Dirac operator

$$D(A) = \gamma^\mu \left(\frac{1}{\imath}\partial_\mu + g A_\mu\right)$$

Index-theorem:
number of zero modes (**i**ndex)=topological charge

$$\nu = \frac{1}{32\pi^2} \int \varepsilon^{\mu\nu\lambda\kappa} \mathrm{tr}\, F_{\mu\nu} F_{\lambda\kappa} d^4x$$

Lattice QCD

- space-time becomes discrete with lattice spacing $\tilde{a}$
- vector field $A_\mu \in \mathrm{su}(3)$ replaced by $U_\mu \in \mathrm{SU}(3)$
- Minkowski space $\rightarrow$ Euclidean space (Wick-rotation $t \rightarrow \imath t$)

The Dirac matrices in the chiral basis

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \rightarrow \gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \rightarrow \gamma^k = \begin{pmatrix} 0 & \imath\sigma^k \\ -\imath\sigma^k & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \rightarrow \gamma^5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

Fundamental problem on the lattice

Energy in continuum:

$$E^2 = k_\mu k^\mu + M_0^2$$

Energy on lattice:

$$E^2 = \frac{\sin^2(k_\mu \widetilde{a})}{\widetilde{a}^2} + M_0^2.$$

Doubler Problem:

$$k_\mu \to \begin{cases} k_\mu \\ \frac{\pi}{\widetilde{a}} - k_\mu \end{cases}$$

one momentum=$(2^4 = 16)$ **particles**

The Wilson-Dirac-operator

Main idea:

- Make 15 particles in the continuum limit ($\widetilde{a} \to 0$) infinitely heavy.

$\Rightarrow$ too inertial, decouple from the system

- Wilson-Dirac operator

$$\begin{aligned} D_{\mathrm{W}} &= D(A) + \widetilde{a}\Delta \\ &\propto \gamma^{\mu}\sin(k_{\mu}\widetilde{a}) + \frac{\sin^2(k_{\mu}\widetilde{a}/2)}{\widetilde{a}} \end{aligned}$$

- Laplace operator Δ
  - additional effective mass
  - explicitly breaks chiral symmetry
  - Dirac operator not anymore anti-Hermitian, but γ_5-Hermitian

$$D_{\mathrm{W}}^{\dagger} = \gamma_5 D_{\mathrm{W}} \gamma_5$$

The ϵ-regime of QCD

- infrared limit of QCD
- lattice volume (space-time volume) $V \to \infty$
- large Compton wavelength of Goldstone bosons
  $\gg$ box size $V^{1/4}$

Saddlepoint approximation

- spontaneous breaking of chiral symmetry
- global Goldstone bosons = Mesons

e.g. $N_\mathrm{f} = 2$, then
$\mathrm{SU}(2)$-integral = zero momentum modes of the three pions

Partition function in ϵ-regime for N_f flavors

$$
\begin{aligned}
Z &\propto \int_{\mathrm{SU}(N_\mathrm{f})} \exp[\mathcal{L}(U)] d\mu(U) \\
&\propto \sum_{\nu\in\mathbb{Z}} \int_{\mathrm{U}\,(N_\mathrm{f})} \exp[\mathcal{L}(U)] \det{}^{\nu} U d\mu(U)
\end{aligned}
$$

Lagrangian of the Goldstone bosons:

$$
\begin{aligned}
\mathcal{L}(U) = \; & \frac{\Sigma V}{2} \mathrm{tr}\,(\widetilde{M}_\mathrm{R} U + U^\dagger \widetilde{M}_\mathrm{L}) \\
& - V W_6 \widetilde{a}^2 [\mathrm{tr}\,(U + U^\dagger)]^2 - V W_7 \widetilde{a}^2 [\mathrm{tr}\,(U - U^\dagger)]^2 \\
& - V W_8 \widetilde{a}^2 \mathrm{tr}\,(U^2 + {U^\dagger}^2)
\end{aligned}
$$

- index of the Dirac operator: ν
- masses for right- and left-handed particles: $\widetilde{M}_{\mathrm{R/L}} = \widetilde{M} \pm \widetilde{\Lambda}$
- lattice spacing: $\widetilde{a}$
- low energy constants: Σ, W_6, W_7, W_8
- spacetime volume: V

a^2-terms of the potential:

$$VW_6\widetilde{a}^2[\mathrm{tr}\,(U+U^\dagger)]^2 + VW_7\widetilde{a}^2[\mathrm{tr}\,(U-U^\dagger)]^2 + VW_8\widetilde{a}^2\mathrm{tr}\,(U^2+{U^\dagger}^2)$$

For $\mathrm{SU}(2)$:

- Sharpe and Singleton (1998)
- Bär, Necco and Schaefer (2009)

For general number of flavors:

- Bär, Rupak and Shoresh (2004)
- Sharpe (2006)

Simplification

$$
\begin{aligned}
& \exp\left[VW_6\widetilde{a}^2[\mathrm{tr}\,(U+U^\dagger)]^2 + VW_7\widetilde{a}^2[\mathrm{tr}\,(U-U^\dagger)]^2\right] \\
= \; & \frac{1}{2\pi}\int d[m_6,\lambda_7]\exp\left[-\frac{{m_6}^2+{\lambda_7}^2}{2}\right] \\
\times \; & \exp\left[\sqrt{VW_6}\widetilde{a}m_6\mathrm{tr}\,(U+U^\dagger) + \sqrt{VW_7}\widetilde{a}\lambda_7\mathrm{tr}\,(U-U^\dagger)\right]
\end{aligned}
$$

- m_6, λ_7 can be considered as additional masses

$\Rightarrow$ omitting the squared trace terms

- can be introduced later on

Random Matrix Ensemble

$$D_{\mathrm{W}} = \begin{pmatrix} aA & W \\ -W^\dagger & aB \end{pmatrix}$$

distributed by

$$P(D_{\mathrm{W}}) \propto \exp\left[-\frac{n}{2}(\mathrm{tr}\, A^2 + \mathrm{tr}\, B^2) - n\, \mathrm{tr}\, WW^\dagger\right]$$

- Hermitian matrices A ($n \times n$) and B ($(n+\nu) \times (n+\nu)$) are the Wilson-terms $\Rightarrow$ breaking of chiral symmetry
- complex W ($n \times (n+\nu)$) matrix
- at $a = 0$: chGUE describing continuum QCD (Shuryak, Verbaarschot; 1993)
- corresponds to $W_8 > 0$

Damgaard, Splittorff, Verbaarschot (2010)

Microscopic Limit

Partition function for N_f flavors

$$Z \propto \int d[D_\mathrm{W}] P(D_\mathrm{W}) \prod_{j=1}^{N_\mathrm{f}} \det(D_\mathrm{W} + m_j \mathbb{1}_{2n+\nu} - \lambda_j \gamma_5)$$

- λ: eigenvalues for $D_5 = D_\mathrm{W}\gamma_5$ with $\gamma_5 = \mathrm{diag}\,(\mathbb{1}_n, -\mathbb{1}_{n+\nu})$
- spacetime volume V / matrix dimension $n \to \infty$
- fixed parameters:
  - $\Sigma V\, \mathrm{diag}\,(\widetilde{M}_\mathrm{R}, \widetilde{M}_\mathrm{L}) = 2n\, \mathrm{diag}\,(m + \lambda, m - \lambda) = \mathrm{diag}\,(\widehat{M}_\mathrm{R}, \widehat{M}_\mathrm{L})$
  - $\sqrt{VW_8}\widetilde{a} = \sqrt{n/2}a = \widehat{a}$

Outcome

$$Z \propto \int_{\mathrm{U}(N_{\mathrm{f}})} \exp[\mathcal{L}(U)] \det{}^{\nu} U d\mu(U)$$

Lagrangian of the Goldstone bosons:

$$\mathcal{L}(U) = \mathrm{tr}\,(\widehat{M}_{\mathrm{R}} U + U^{\dagger} \widehat{M}_{\mathrm{L}}) - \widehat{a}^2 \mathrm{tr}\,(U^2 + {U^{\dagger}}^2)$$

Damgaard, Splittorff, Verbaarschot (2010)

The Joint Probability Densities

Properties of $D_{\rm W}$ and D_5

- $D_{\rm W}$ is γ_5-Hermitian $\Leftrightarrow$ $D_5 = D_{\rm W}\gamma_5$ is Hermitian
- form invariance:
  $D_5:\ V \in {\rm U}(2n+\nu),\ V^{-1} = V^\dagger$, compact
  $D_{\rm W}:\ U \in {\rm U}(n, n+\nu),\ U^{-1} = \gamma_5 U^\dagger \gamma_5$, non-compact
- diagonalization:

$$
\begin{aligned}
D_5 &= VxV^{-1} \\
D_{\rm W} &= U \left(\begin{array}{cc|cc} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ \hline 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{array}\right) U^{-1}
\end{aligned}
$$

x, x_j, y_2 are real diagonal

Diagonalization of $D_{\rm W}$ and D_5

- $$D_5 = VxV^{-1}$$

  x: $2n+\nu$ dim $\Rightarrow$ pure real spectrum

- Let $0 \leq l \leq n$

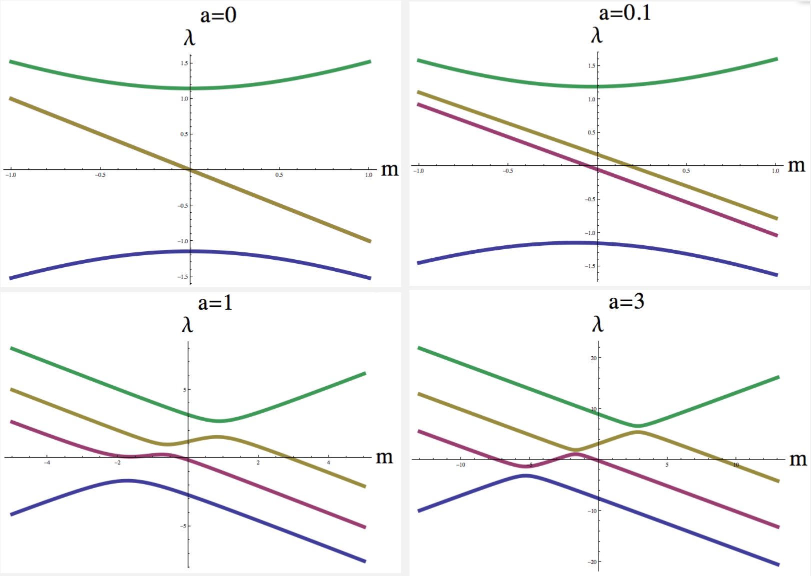
  $$D_{\rm W} \quad = \quad U \left(\begin{array}{cc|cc} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ \hline 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{array}\right) U^{-1}$$

  x_1: l dim $\Rightarrow$ real spectrum

  x_2, y_2: $n-l$ dim $\Rightarrow$ complex conj. eigenvalue pairs $x_2 \pm \imath y_2$

  x_3: $\nu + l$ dim $\Rightarrow$ real spectrum

Spectral flow of D_5 (Example for $n = 1$ and $\nu = 2$)



Definition of the Joint Probability Density

Let f be arbitrary integrable function invariant under $\mathrm{Gl}(2n+\nu,\mathbb{C})$:

$$\begin{aligned}\int f(D_5)P(D_\mathrm{W})d[D_\mathrm{W}] &= \int f(x)p_5(x)d[x]\\ \int f(D_\mathrm{W})P(D_\mathrm{W})d[D_\mathrm{W}] &= \sum_{l=0}^{n}\int f(z^{(l)})p_\mathrm{W}^{(l)}(z^{(l)})d[z^{(l)}]\\ &= \int f(z)p_\mathrm{W}(z)d[z]\end{aligned}$$

where

$$\begin{aligned}p_\mathrm{W}(z) &= \sum_{l=0}^{n}\int p_\mathrm{W}^{(l)}(z^{(l)})\delta\left(z-z^{(l)}\right)d[z^{(l)}],\\ z^{(l)} &= \mathrm{diag}\left(x_1, x_2+\imath y_2, x_2-\imath y_2, x_3\right)\end{aligned}$$

Results

For D_5 a $2(n+\nu)$ dim Pfaffian:

$$p_5(x) \propto \Delta_{2n+\nu}(x)\mathrm{Pf}\begin{bmatrix} g_2(x_a, x_b) & x_a^{b-1} g_1(x_a) \\ -x_b^{a-1} g_1(x_b) & 0 \end{bmatrix}$$

degenerated quark mass m
Akemann, Nagao (2011)

For D_W a $n+\nu$ dim determinant:

$$\begin{aligned} & p_\mathrm{W}(z) \propto \Delta_{2n+\nu}(z) \\ \times \quad & \det\begin{bmatrix} g_\mathrm{c}(z_{a\mathrm{R}})\delta^{(2)}(z_{a\mathrm{R}} - z_{b\mathrm{L}}^*) + g_\mathrm{r}(x_{a\mathrm{R}}, x_{b\mathrm{L}})\delta(y_{a\mathrm{R}})\delta(y_{b\mathrm{L}}) \\ x_{b\mathrm{L}}^{a-1} g_1(x_{b\mathrm{L}})\delta(y_{b\mathrm{L}}) \end{bmatrix} \end{aligned}$$

degenerated source term $\lambda\gamma_5$
Kieburg, Verbaarschot, Zafeiropoulos (2011)

Remark: Vandermonde determinant $\Delta_{2n+\nu}(x) = \prod_{i<j}(x_i - x_j)$

Side remark: Orthogonal Polynomial Theory

A novel mixing of orthogonal and skew-orthogonal polynomials

- orthogonal polynomials from order 0 to $\nu - 1$

$$\langle p_i | p_j \rangle \propto \delta_{ij}$$

- skew-orthogonal polynomials from order ν

$$\begin{aligned} (q_{\nu+2i} | q_{\nu+2j+1}) &\propto \delta_{ij} \\ (q_{\nu+2i} | q_{\nu+2j}) &= (q_{\nu+2i+1} | q_{\nu+2j+1}) = 0 \end{aligned}$$

- additional relations

$$(p_i | p_j) = (p_i | q_{\nu+j}) = \langle p_i | q_{\nu+j} \rangle = 0$$

with

$$\begin{aligned} \langle f_1 | f_2 \rangle &= \int f_1(x) f_2(x) g_1(x) dx \\ (f_1 | f_2) &= \int (f_1(x_1) f_2(x_2) - f_1(x_2) f_2(x_1)) G_2(x_1, x_2) d[x] \end{aligned}$$

- essentially the same system of equations for D_{W}

Side remark: Orthogonal Polynomial Theory

Wish list:

- relation to Hermite polynomials
- recursion relation
- Christoffel Darboux-like formula
- representation as a matrix integral
- Rodrigues formula
- asymptotics in the microscopic limit

The Eigenvalue Densities

Definition

For D_5

$$\rho_5(x_1) = \int p_5(x) d[x_{\neq 1}]$$

$\Rightarrow$ one level density ρ_5

For $D_{\rm W}$

$$\begin{aligned} \rho_{\rm R}(x_{1\rm R})\delta(y_{1\rm R}) + \frac{1}{2}\rho_{\rm c}(z_{1\rm R}) &= \int p_{\rm W}(z) d[z_{\neq 1\rm R}] \\ \rho_{\rm L}(x_{1\rm L})\delta(y_{1\rm L}) + \frac{1}{2}\rho_{\rm c}(z_{1\rm L}) &= \int p_{\rm W}(z) d[z_{\neq 1\rm L}] \end{aligned}$$

$\Rightarrow$ three level densities $\rho_{\rm c}$ and

$$\begin{aligned} \rho_{\rm r} &= 2\rho_{\rm R} = \rho_{\rm R} + \rho_{\rm L} - \rho_\chi \\ \rho_\chi &= \rho_{\rm R} - \rho_{\rm L} \end{aligned}$$

In the Microscopic Limit

For $\widehat{a} = 0$:

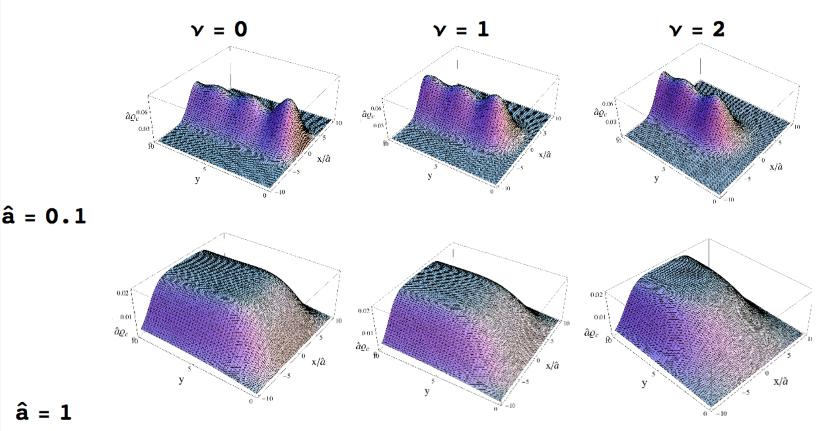
$$\begin{aligned} \rho(y) &= \nu\delta(y) + \frac{y}{2}(J_\nu^2(y) - J_{\nu-1}(y)J_{\nu-2}(y)) \\ &= \nu\delta(y) + \rho_{\rm i}^{(\nu)}(y) \end{aligned}$$

For $\widehat{a} \neq 0$:
two-fold integrals for $\rho_5, \rho_{\rm c}, \rho_{\rm r}, \rho_\chi$

Akemann, Damgaard, Kieburg, Nagao, Splittorff, Verbaarschot, Zafeiropoulos (2010/11)

The Density ρ_c

$$\rho_c(z) = \begin{cases} \dfrac{\exp[-x^2/8\hat{a}^2]}{\sqrt{8\pi}\hat{a}} \dfrac{|y|}{|z|} \rho_i^{(\nu)}(|z|), & \hat{a} \ll 1 \\ \dfrac{\Theta(8\hat{a}^2 - |x|)}{16\pi\hat{a}^2} \mathrm{erf}\left[\dfrac{|y|}{\sqrt{8}\hat{a}}\right], & \hat{a} \gg 1 \end{cases}$$

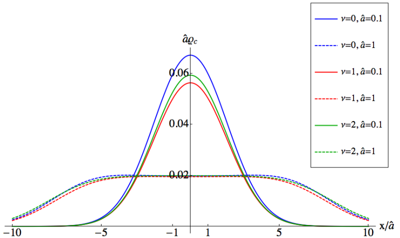


Kieburg, Verbaarschot, Zafeiropoulos (2011)

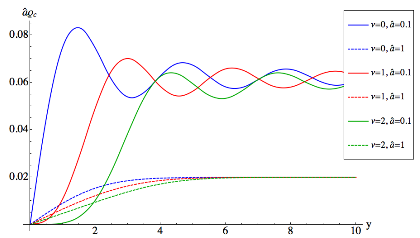
The Density $\rho_{\rm c}$

$$\rho_{\rm c}(z) = \begin{cases} \dfrac{\exp[-x^2/8\hat{a}^2]}{\sqrt{8\pi}\hat{a}} \dfrac{|y|}{|z|} \rho_{\rm i}^{(\nu)}(|z|), & \hat{a} \ll 1 \\ \dfrac{\Theta(8\hat{a}^2 - |x|)}{16\pi\hat{a}^2} \mathrm{erf}\left[\dfrac{|y|}{\sqrt{8}\hat{a}}\right], & \hat{a} \gg 1 \end{cases}$$

along $y = 5\hat{a}$ axis



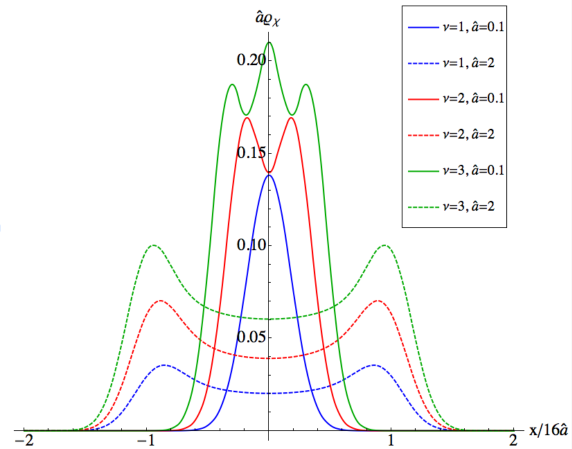
along imaginary axis



Kieburg, Verbaarschot, Zafeiropoulos (2011)

The Density ρ_χ

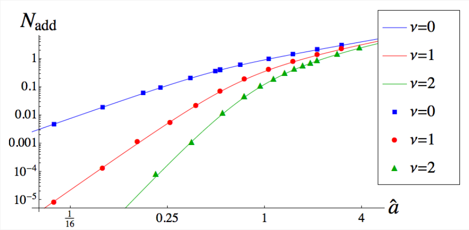
$$\rho_\chi(x) = \begin{cases} \rho^{(\nu)}_{\text{GUE}}\left(\dfrac{x}{4\hat{a}}\right), & \hat{a} \ll 1 \\ \dfrac{\nu\Theta(8\hat{a}^2 - |x|)}{\pi\sqrt{64\hat{a}^4 - x^2}}, & \hat{a} \gg 1 \end{cases}$$



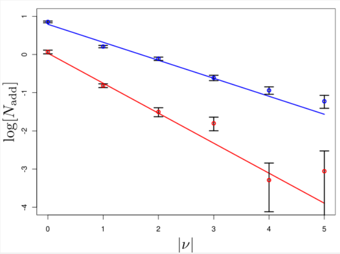
Akemann, Damgaard, Splittorff, Verbaarschot (2010/11)

Number of additional real modes $N_{\rm add}$

$$N_{\rm add} = \int \rho_{\rm r}(x)dx = \begin{cases} \dfrac{[2(\nu+1)]!}{[(\nu+1)!]^2}\left(\dfrac{\hat{a}^2}{2}\right)^{\nu+1}, & \hat{a} \ll 1 \\ \left(\dfrac{2}{\pi}\right)^{3/2}\hat{a}, & \hat{a} \gg 1 \end{cases}$$



Kieburg, Verbaarschot, Zafeiropoulos (2011)



Deuzeman, Wenger, Wuilloud (2011)

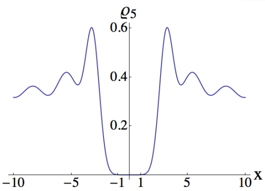
ρ_5 at small Lattice Spacing

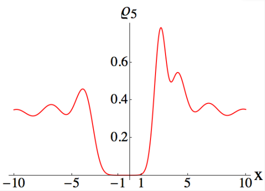
$$\rho_5(x) = \rho^{(\nu)}_{\mathrm{GUE}}\left(\frac{x-\hat{m}}{4\hat{a}}\right) + \frac{|x|}{\sqrt{x^2-\hat{m}^2}}\rho^{(\nu)}_{\mathrm{i}}(\sqrt{x^2-\hat{m}^2})$$

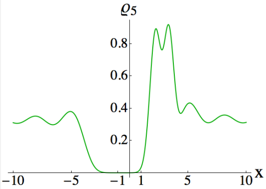
$$= \rho_\chi(x-\hat{m}) + \sqrt{8\pi}\hat{a}\exp\left[-\frac{\hat{m}^2}{8\hat{a}^2}\right]\rho_{\mathrm{c}}(\mathrm{Re} = \imath\hat{m}, \mathrm{Im} = x)$$

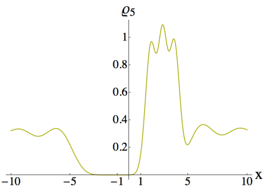
$\hat{m} = 3$, $\hat{a} = 0.2$

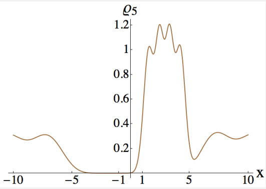
$\nu \in \{0, 1, 2, 3, 4\}$











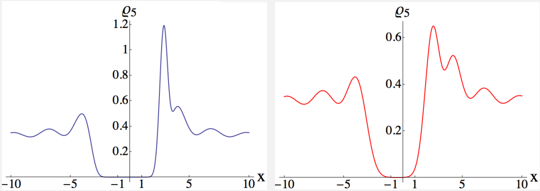
Akemann, Damgaard, Splittorff, Verbaarschot (2010/11)

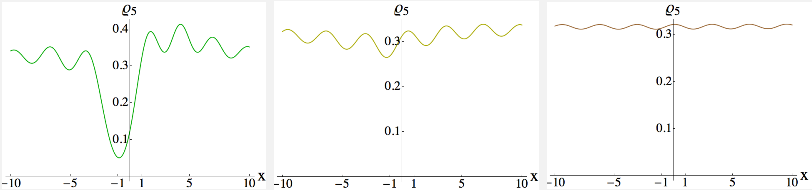
Lattice Spacing Dependence of ρ_5

$$\rho_5(x) = \begin{cases} \rho^{(\nu)}_{\mathrm{GUE}}\left(\dfrac{x-\hat{m}}{4\hat{a}}\right) + \dfrac{|x|}{\sqrt{x^2-\hat{m}^2}}\rho^{(\nu)}_{\mathrm{i}}(\sqrt{x^2-\hat{m}^2}), & \hat{a} \ll 1 \\ \dfrac{1}{\pi}\left[1+\dfrac{\cos^2 x}{8\hat{a}^2}\right], & \hat{a} \gg 1 \end{cases}$$

$\hat{m} = 3$, $\nu = 1$

$\hat{a} \in \{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2\}$

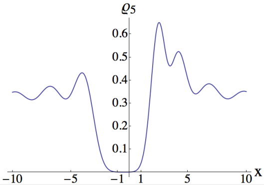




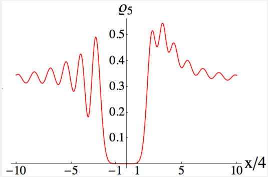
Mean Field Limit of ρ_5

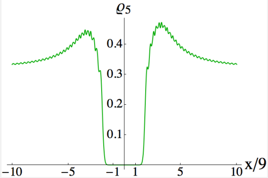
Scaling:

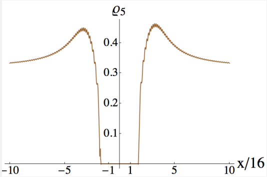
- $\widehat{m} \to s^2 \widehat{m}$
- $x \to s^2 x$
- $\widehat{a} \to s\widehat{a}$



$\widehat{m} = 3$, $\nu = 1$, $\widehat{a} = 0.25$
$s \in \{1, 2, 3, 4\}$



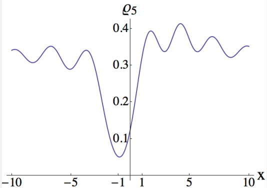




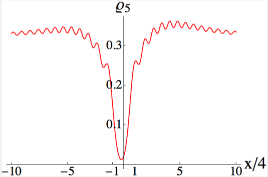
Mean Field Limit of ρ_5

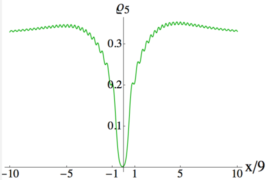
Scaling:

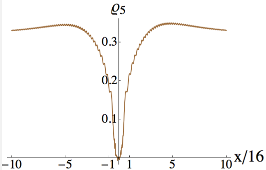
- $\hat{m} \to s^2 \hat{m}$
- $x \to s^2 x$
- $\hat{a} \to s\hat{a}$



$\hat{m} = 3,\ \nu = 1,\ \hat{a} = 0.5$
$s \in \{1, 2, 3, 4\}$





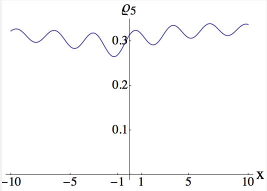


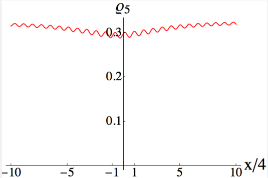
Mean Field Limit of ρ_5

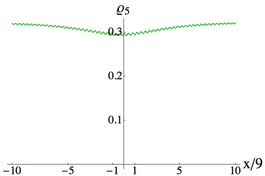
Scaling:

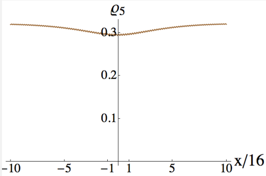
- $\widehat{m} \to s^2 \widehat{m}$
- $x \to s^2 x$
- $\widehat{a} \to s\widehat{a}$

$\widehat{m} = 3, \nu = 1, \widehat{a} = 1$
$s \in \{1, 2, 3, 4\}$









ν Dependence of ρ_5 at large Lattice Spacing

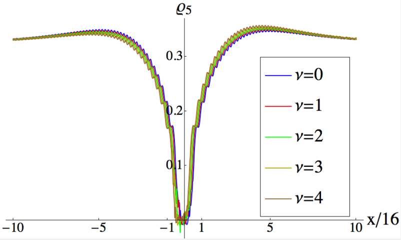
Scaling:

- $\widehat{m} \to s^2 \widehat{m}$
- $x \to s^2 x$
- $\widehat{a} \to s\widehat{a}$

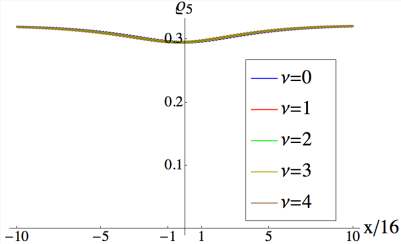
$\widehat{m} = 3$, $\widehat{a} \in \{0.5, 1\}$, $s = 4$

gap in $|x| \leq \left(\widehat{m}^{2/3} - 4\widehat{a}^{4/3}\right)^{3/2}$
$\Rightarrow$ gap only if $\widehat{m} > 8\widehat{a}^2$

not Aoki Phase



Aoki Phase



Damgaard, Splittorff, Verbaarschot (2010)

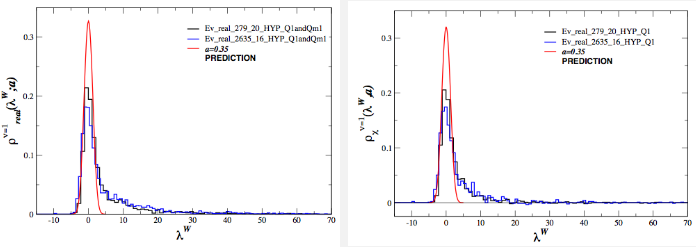
Conclusions and Outlook

Summary

- joint probability densities of D_{W} and D_5
- small $\widehat{a} \ll 1$:
  - broadening of ν formerly zero modes by GUE
  - Gaussian broadening of $\rho_{\mathrm{i}}(y)$ for D_{W}
  - additional real modes are strongly suppressed with increasing ν
- large $\widehat{a} \gg 1$:
  - finite support of size $\approx \widehat{a}^2$ along and parrallel to the real axis for D_{W}
  - ν independence of D_{W} and D_5
  - mean field limit: gap for ρ_5 if $\widehat{m} > 8\widehat{a}^2$

Outlook

- non-degenerated masses
- higher correlation functions
- comparison with lattice data



Damgaard, Heller, Splittorff (2011)

- single eigenvalue distributions

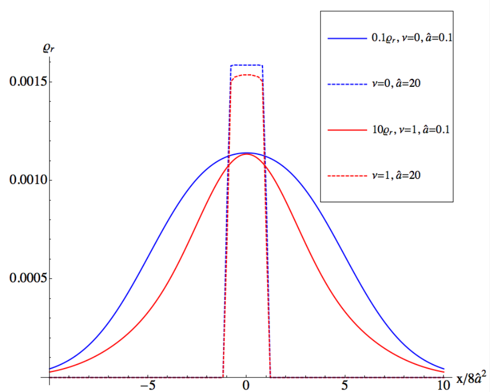
Thank you for your attention!

Collaborators:
Gernot Akemann
Poul H. Damgaard
Kim Splittorff
Jacobus J. M. Verbaarschot
Savvas Zafeiropoulos

Appendix

The Density $\rho_{\rm r}$

$$\rho_{\rm r}(x) = \begin{cases} \hat{a}^{2\nu+1} \int_1^{\sqrt{2}} p_\nu\left(\frac{x^2}{\hat{a}^2}, \lambda^2 - 1\right) \exp\left[-\frac{x^2}{16\hat{a}^2}\lambda^2\right] d\lambda, & \hat{a} \ll 1 \\ \frac{\Theta(8\hat{a}^2 - |x|)}{(2\pi)^{3/2} 2\hat{a}}, & \hat{a} \gg 1 \end{cases}$$



Kieburg, Verbaarschot, Zafeiropoulos (2011)