"Supersymmetry without Supersymmetry": Determinants and Pfaffians in RMT

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Characteristic polynomials

Characteristic polynomial of a $N \times N$ matrix H is

$$q(x,H) = \det(H - x) \quad , \quad x \in \mathbb{C}$$

Properties

- q is a polynomial of order N in x
- q is invariant under similarity transformations

$$H \longrightarrow THT^{-1} \quad \text{with } T \text{ invertible}$$

- roots of q with respect to x are the algebraic eigenvalues $\{E_1, \ldots, E_N\}$ of H

$\Rightarrow$ q only depends on x and $\{E_1, \ldots, E_N\}$

Average over rotation invariant matrix ensembles are interesting for:

- disordered systems
- quantum chaos
- matrix models in high energy physics
- quantum chromodynamics
- quantum gravity
- econo physics
- number theory
- Weyl's character formula
- theory of orthogonal polynomials

SETTING

$$Z(\kappa,\lambda) \sim \int P(H) \frac{\prod\limits_{n=1}^{k_2} \det(H-\kappa_{n2}) \prod\limits_{m=1}^{l_2} \det(H^\dagger-\lambda_{m2})}{\prod\limits_{n=1}^{k_1} \det(H-\kappa_{n1}) \prod\limits_{m=1}^{l_1} \det(H^\dagger-\lambda_{m1})} d[H]$$

+ factorizable probability density P

$$P(E) = \prod_{j=1}^{N} \widetilde{P}(E_j) \quad , \text{ with } E = \mathrm{diag}\,(E_1,\ldots,E_N)$$

H does not have to be symmetric!

First example:

Hermitian matrices

$\mathrm{Herm}\,(N)$

STARTING POINT

$$Z_N(\kappa) = \int\limits_{\mathrm{Herm}\,(N)} P(H) \prod_{j=1}^{k} \frac{\det(H - \kappa_{j2})}{\det(H - \kappa_{j1})} d[H]$$

It is well known that this generating function exhibits a determinantal structure!
Baik, Deift, Strahov (2003); Grönqvist, Guhr, Kohler (2004); Borodin, Strahov (2005); Guhr (2006)

Similar determinantal structures for the k-point correlation function.
Mehta,Gaudin (1960)

First example: Hermitian matrices $\mathrm{Herm}(N)$

Definition

$$\sqrt{\mathrm{Ber}_{p/q}(s)} = \frac{\prod\limits_{1\leq a<b\leq p}(s_{a1}-s_{b1})\prod\limits_{1\leq a<b\leq q}(s_{a2}-s_{b2})}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(s_{a1}-s_{b2})} = \frac{\Delta_p(s_1)\Delta_q(s_2)}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(s_{a1}-s_{b2})}$$

$$s = \mathrm{diag}(s_1, s_2) = \mathrm{diag}(s_{11}, \ldots, s_{p1}, s_{12}, \ldots, s_{q2})$$

RESULT

$$Z_N(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \det\left[\frac{Z_N(\kappa_{a1}, \kappa_{b2})}{\kappa_{a1} - \kappa_{b2}}\right]$$

We have not used the explicit form of $\widetilde{P}(E)$!
$\longrightarrow$ These structures have to be true for other matrix ensembles!

Determinantal structures
and
the connection to supersymmetry

HERMITIAN SUPERMATRICES

- symmetric supermatrices with respect to the supergroup $U(p/q)$

$$\sigma \in \left[\begin{array}{c|c} \mathrm{Herm}\,(p) & [pq\mathrm{G.v.}]^\dagger \\ \hline pq\mathrm{G.v.} & \mathrm{Herm}\,(q) \end{array}\right] \quad ; \quad \sigma = \sigma^\dagger$$

- diagonalization:

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_\eta^\dagger \\ \sigma_\eta & \sigma_2 \end{bmatrix} \longrightarrow s = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \quad , \quad \sigma = UsU^\dagger$$

$\Rightarrow$ measure for the eigenvalues:

$$d[\sigma] \longrightarrow \mathrm{Ber}_{p/q}(s) d[s]$$

Determinantal structures and the connection to supersymmetry ($p \leq q$)

(1)

$$\sqrt{\mathrm{Ber}_{p/q}(s)} \sim \det \overbrace{\begin{bmatrix} \dfrac{1}{s_{a1} - s_{b2}} \\ s_{b2}^{a-1} \end{bmatrix}}^{q} \begin{matrix} \}p \\ \}q-p \end{matrix}$$

mixes “Cauchy–terms” with “Vandermonde–terms”

(2)

$$\sqrt{\mathrm{Ber}_{p/q}(s)} \sim \det \begin{bmatrix} \left(\dfrac{s_{a1} - \imath\varepsilon}{s_{b2} - \imath\varepsilon}\right)^{p-q} \dfrac{1}{s_{a1} - s_{b2}} \\ s_{b2}^{a-1} \end{bmatrix}$$

true for arbitrary ε, useful for supersymmetric calculations

$^{(1)}$ Basor, Forrester[94] without considering the connection to supersymmetry;

$^{(1),(2)}$ Kieburg, Guhr[09] exhibiting the intimate relation to supersymmetry

Second example:
Special orthogonal group
$\mathrm{SO}(2N)$

Second example: Special orthogonal group $\mathrm{SO}(2N)$

Starting point

$$Z_{2N}(\kappa) = \int_{\mathrm{SO}(2N)} \prod_{j=1}^{k} \frac{\det(O - \kappa_{j2})}{\det(O - \kappa_{j1})} d\mu(O)$$

$d\mu(O)$ normalized Haar measure on $\mathrm{SO}(2N)$

Interesting for mathematicians: Weyl–type character formula

Huckleberry, Püttmann, Zirnbauer (2005)

Second example: Special orthogonal group $\mathrm{SO}(2N)$

Result

$$Z_{2N}(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\tilde{\kappa})}} \det\left[\frac{Z_{2N}(\kappa_{a1},\kappa_{b2})}{\tilde{\kappa}_{a1}-\tilde{\kappa}_{b2}}\right]$$

We have not used the explicit form of $\widetilde{P}(E)$!

Similar results can be found for $\mathrm{SO}(2N+1)$, for special orthogonal groups times reflections, for $\mathrm{USp}(2N)$, for their Lie algebras and etc.

Aim of our method

Two important questions

Do all determinantal and Pfaffian structures have the same origin?

If yes:

What are the conditions to find such structures?

REASON OF DETERMINANTAL AND PFAFFIAN STRUCTURES

The integrand already contains a determinantal structure including the characteristic polynomials!

Basor and Forrester: CUE by algebraic rearrangement (1994)
Our method considerably extends this idea and makes the connection to supersymmetry!

Our aim

Example for determinantal structures (schematic)

$$Z(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \det\left[\frac{Z(\kappa_{a2}, \kappa_{b1})}{\kappa_{a2} - \kappa_{b1}}\right]$$

Generating functions for k-point correlations can be expressed by generating functions corresponding to one-point correlation functions!

For Pfaffian structures, we find simplifications similar to this formula!

The structures carry over to the large N limit!

Theorem for determinantal structures

Starting point

$$Z_N(\kappa,\lambda) \sim \int \prod_{j=1}^{N} \widetilde{P}(E_j) \frac{\prod_{n=1}^{k_2}(E_j-\kappa_{n2}) \prod_{m=1}^{l_2}(E_j^*-\lambda_{m2})}{\prod_{n=1}^{k_1}(E_j-\kappa_{n1}) \prod_{m=1}^{l_1}(E_j^*-\lambda_{m1})} |\Delta_N(E)|^2 d[E]$$

Result for $d = N + k_2 - k_1 = N + l_2 - l_1 \geq 0$

$$Z_N(\kappa,\lambda) \sim \frac{\det\left[\begin{array}{c|c} Z_{d-1}(\kappa_{b2},\lambda_{a2}) & \dfrac{Z_d(\lambda_{b1},\lambda_{a2})}{\lambda_{b1}-\lambda_{a2}} \\ \hline \dfrac{Z_d(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}} & Z_{d+1}(\kappa_{a1},\lambda_{b1}) \end{array}\right]}{\sqrt{\mathrm{Ber}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}_{l_1/l_2}(\lambda)}}$$

STARTING POINT

$$Z_N(\kappa) \sim \int \prod_{j=1}^{N} g(E_{2j-1}, E_{2j}) \prod_{j=1}^{2N} \frac{\prod_{n=1}^{k_2} (E_j - \kappa_{n2})}{\prod_{n=1}^{k_1} (E_j - \kappa_{n1})} \Delta_{2N}(E) d[E]$$

RESULT FOR AN INTEGER $d = N + (k_2 - k_1)/2 \geq 0$

$$Z_N(\kappa) \sim \frac{\mathrm{Pf}\left[\begin{array}{c|c} (\kappa_{b2} - \kappa_{a2}) Z_{d-1}(\kappa_{a2}, \kappa_{b2}) & \dfrac{Z_d(\kappa_{b1}, \kappa_{a2})}{\kappa_{b1} - \kappa_{a2}} \\ \hline \dfrac{Z_d(\kappa_{a1}, \kappa_{b2})}{\kappa_{b2} - \kappa_{a1}} & (\kappa_{b1} - \kappa_{a1}) Z_{d+1}(\kappa_{a1}, \kappa_{b1}) \end{array}\right]}{\sqrt{\mathrm{Ber}_{k_1/k_2}(\kappa)}}$$

General procedure

(1) algebraic rearrangement

(2) determinantal (supersymmetric) structure

(3) integration theorems

(4) Leibniz expansion

(5) identification of the kernels

General remarks and conclusions

FOR PFAFFIAN AND DETERMINANTAL STRUCTURES

- structures result from an algebraic construction
- the joint probability density has only to factorize and the integrals have to be finite, **no other requirements**

$\Rightarrow$ applicable to a broad class of ensembles

SOME MATRIX ENSEMBLES YIELDING DETERMINANTS

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $g(z)$
Hermitian ensemble [57, 31, 62, 32, 34, 35]	$\widetilde{P}(\operatorname{tr} H^m, m \in \mathbb{N})$ $H = H^\dagger$	H	$P(x)\delta(y)$
circular unitary ensemble (unitary group) [37, 63, 14, 13, 38, 20, 64, 21]	$\widetilde{P}(\operatorname{tr} U^m, m \in \mathbb{N})$ $U^\dagger U = \mathbb{1}_N$	U and $U^\dagger$	$P(e^{\imath\varphi})\,\delta(r-1)$
Hermitian chiral (complex Laguerre) ensemble [65, 66, 67, 7]	$\widetilde{P}(\operatorname{tr}(AA^\dagger)^m, m \in \mathbb{N})$ A is a complex $N \times M$ matrix with $N \leq M$	$AA^\dagger$	$P(x)x^{M-N}\Theta(x)\delta(y)$
Gaussian elliptical ensemble [9, 10, 11, 36]; for $\tau = 1$ complex Ginibre ensemble	$\exp\left[-\frac{(\tau+1)}{2}\operatorname{tr} H^\dagger H\right] \times$ $\times \exp\left[-\frac{(\tau-1)}{2}\mathrm{Re}\ \operatorname{tr} H^2\right]$ H is a complex matrix; $\tau > 0$	H and $H^\dagger$	$\exp\left[-r^2\left(\sin^2\varphi + \tau\cos^2\varphi\right)\right]$
Gaussian complex chiral ensemble [12]	$\exp\left[-\operatorname{tr} A^\dagger A - \operatorname{tr} B^\dagger B\right]$ $C = \imath A + \mu B$ $D = \imath A^\dagger + \mu B^\dagger$ A and B are complex $N \times M$ matrices with $N \leq M$	CD and $D^\dagger C^\dagger$	$K_{M-N}\left(\frac{1+\mu^2}{2\mu^2}r\right)r^{M-N}\times$ $\times \exp\left(\frac{1-\mu^2}{2\mu^2}r\cos\varphi\right)$

SOME MATRIX ENSEMBLES YIELDING PFAFFIAN STRUCTURES

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability densities $g(z_1, z_2)$ and $\tilde{g}(z_1, z_2)$	probability density $h(z)$
real symmetric matrices [31, 24, 18]	$\widetilde{P}\left(\mathrm{tr}\, H^m, m \in \mathbb{N}\right)$ $H = H^T = H^*$	H	$P(x_1)P(x_2)\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)$	$P(x)\delta(y)$
circular orthogonal ensemble [4]	$\widetilde{P}\left(\mathrm{tr}\, U^m, m \in \mathbb{N}\right)$ $U^\dagger U = 1\!\!1_N$ and $U^T = U$	U and $U^\dagger$	$P(e^{\imath\varphi_1})P(e^{\imath\varphi_2})\times$ $\times\delta(r_1 - 1)\delta(r_2 - 1)\times$ $\times\Theta(\varphi_2 - \varphi_1)$	$P(e^{\imath\varphi})\delta(r - 1)$
real symmetric chiral (real Laguerre) ensemble [21, 32, 33, 34]	$\widetilde{P}\left(\mathrm{tr}(AA^T)^m, m \in \mathbb{N}\right)$ A is a real $N \times M$ matrix with $\nu = M - N \geq 0$	AA^T	$P(x_1)P(x_2)\times$ $\times(x_1x_2)^{(\nu-1)/2}\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)$	$P(x)\delta(y)x^{(\nu-1)/2}$
Gaussian real elliptical ensemble; for $\tau = 1$ real Ginibre ensemble [10, 15, 35, 16, 36, 25] [37, 38, 23]	$\exp\left[-\frac{(\tau+1)}{2}\,\mathrm{tr}\, H^T H\right]\times$ $\times\exp\left[-\frac{(\tau-1)}{2}\,\mathrm{tr}\, H^2\right]$ $H = H^*$; $\tau > 0$	H	$\prod\limits_{j\in\{1,2\}}\exp\left[-\tau x_j^2\right]\times$ $\times\sqrt{\mathrm{erfc}(\sqrt{2(1+\tau)}y_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)+$ $+2\imath\delta^2(z_1 - z_2^*)\Theta(y_1)]$	$\exp(-\tau x^2)\delta(y)$
Gaussian real chiral ensemble [9, 39]	$\exp\left[-\,\mathrm{tr}\, A^T A - \mathrm{tr}\, B^T B\right]$ $C = A + \mu B$ $D = -A^T + \mu B^T$ A and B are real $N \times M$ matrices with $\nu = M - N \geq 0$	CD	$\prod\limits_{j\in\{1,2\}}\exp\left[-2\eta_- z_j\right]\times$ $\times\lvert z_j\rvert^\nu\sqrt{f(2\eta_+ z_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2 - x_1)+$ $+2\imath\delta^2(z_1 - z_2^*)\Theta(y_1)]$	$x^{\nu/2}\exp\left[-2\eta_- x\right]\times$ $\times K_{\nu/2}(2\eta_+ x)\delta(y)$

FOR MANY CHARACTERISTIC FUNCTIONS IN THE DENOMINATOR

- determinantal structures for $k_1 - k_2 - N$ and $l_1 - l_2 - N \geq 0$:

$$Z_N(\kappa, \lambda) \sim \frac{1}{\sqrt{\mathrm{Ber}_{k_1/k_2}(\kappa)}\sqrt{\mathrm{Ber}_{l_1/l_2}(\lambda)}} \times \det \left[\begin{array}{c|c|c} 0 & 0 & \dfrac{1}{\lambda_{b1} - \lambda_{a2}} \\ \hline 0 & 0 & \lambda_{b1}^{a-1} \\ \hline \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & \kappa_{a1}^{b-1} & Z_2(\kappa_{a1}, \lambda_{b1}) \end{array} \right]$$

- similar results are true for Pfaffian structures

$\Rightarrow$ two regimes depending on the number of characteristic polynomials

General remarks and conclusions

Relation to orthogonal polynomial method

- supersymmetric structures are the ultimate reason for the Dyson-Mehta-Mahoux integration theorem
- determinantal and Pfaffian structures lead to orthogonality and skew-orthogonality relations

$\Rightarrow$ orthogonal and skew-orthogonal polynomials and their Cauchy-transforms have simple expressions as matrix averages
(Akemann, Kieburg, Phillips; in progress)

For Pfaffian structures

Real and quaternionic ensembles are not distinguishable because their Pfaffian structures have the same origin!

General remarks and conclusions

Other applications

- ensembles in the presence of an external field and intermediate ensembles
- all Efetov–Wegner terms for the supersymmetric Itzykson–Zuber integral
- supersymmetry calculations in general

OTHER APPLICATIONS

We use structures of supersymmetry without ever mapping our integrals onto superspace!

$\Rightarrow$**Supersymmetry without supersymmetry**

THANK YOU FOR YOUR ATTENTION!

- M. Kieburg and T. Guhr. "Derivation of determinantal structures for random matrix ensembles in a new way" *J. Phys.* **A 43** 075201 (2010)
  preprint: arXiv:0912.0654
- M. Kieburg and T. Guhr. "A new approach to derive Pfaffian structures for random matrix ensembles"
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