Arbitrary rotation invariant random matrix ensembles: Hubbard-Stratonovitch transformation versus superbosonization

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- approach for Gaussian matrix ensembles by Efetov et al. (80's)

Mathematical question: Is the supersymmetry method restricted to Gaussian matrix ensemles?

- proof of the universality on the local scale by Hackenbroich and Weidenmueller (1995); Dirac–distribution
- first approaches by
  - Lehmann, Saher, Sokolov, and Sommers: chaotic scattering (1995); Dirac–Distribution
  - Efetov, Schwiete, and Takahashi: disordered and chaotic systems (2004); notion of superbosonisation
  - etc.

Introduction

Generalized Hubbard–Stratonovich transformation

arbitrary rotation invariant ensembles on symmetric matrices

- unitary symmetry: Guhr (2006); partly heuristic
- orthogonal and unitary symplectic symmetry: Kieburg, Grönqvist, and Guhr (2008); mathematically more rigorous

Superbosonization Formula

arbitrary rotation invariant ensembles on symmetric matrices

- orthogonal symmetry: Sommers (2007); idea of the proof
- of all symmetry classes: Littelmann, Sommers, and Zirnbauer (2007/2008); mathematically rigorous

Comparison of both approaches

Kieburg, Sommers, and Guhr (2009); for all symmetry classes

Limitation of the method of orthogonal polynomials:

- factorizable probability density P

$$P(H) \sim \det f(H) \quad , \text{ with } \quad f(AHA^{-1}) = Af(H)A^{-1}$$

Example

Gaussian matrix ensembles $P(H) \sim e^{-\mathrm{tr}\,(H^2)}$

- sufficient on the local scale $\Rightarrow$ universality

Arbitrary rotation invariant matrix ensembles

- non-factorizable ensembles
- proof of universality on the local mean level spacing in a mathematically rigorous way (requirements of P)
- financial correlation matrices
- high-energy physics and quantum gravity: $P(H) \sim \exp(-\sum_j c_j \operatorname{tr} H^j)$
- fundamental mathematical question: supersymmetry for non-Gaussian ensembles

Average over rotation invariant matrix ensembles are interesting for:

- disordered systems
- quantum chaos
- number theory
- matrix models in high energy physics
- etc.

$$Z(x^-) \sim \int P(H) \frac{\prod_{n=1}^{q} \det^{\lambda_2}(H - x_{n2}^-)}{\prod_{m=1}^{p} \det^{\lambda_1}(H - x_{m1}^-)} d[H]$$

$$\sim \int P(E) \prod_{l=1}^{N} \frac{\prod_{n=1}^{q} (E_l - x_{n2}^-)}{\prod_{m=1}^{p} (E_l - x_{m1}^-)^{\beta/2}} |\Delta(E)|^{\beta} d[E]$$

Dyson index $\beta \in \{1, 2, 4\}$

Map from ordinary space to superspace such that

$$Z(x^-) \sim \int Q(\sigma)\mathrm{Sdet}^{-N/\gamma}(\sigma - x^-)d[\sigma]$$

or in Fourier space

$$\begin{aligned} Z(x^-) &\sim \int \Phi_0(\rho) I(\rho) \exp(-\imath \mathrm{Str}\, \rho x^-) d[\rho] \\ &\sim \int \Phi_0(\rho) \mathrm{Sdet}^{\kappa} \rho \exp(-\imath \mathrm{Str}\, \rho x^-) d[\rho] \end{aligned}$$

- $Q = \mathcal{F}\Phi_0$ probability density in superspace
- Φ_0 is "characteristic function" of P **and** Q

$$\frac{1}{\det(H-x_{m1}^{-})^{\lambda_1}} \sim \int \exp\left[\imath z_m^\dagger (H - x_{m1}^{-}) z_m\right] d[z_m]$$

vectors of ordinary variables z_m: $[z_{ma}, z_{mb}]_- = 0$

$$\det(H-x_{n2}^{-})^{\lambda_2} \sim \int \exp\left[\imath \eta_n^\dagger (H - x_{n2}^{-}) \eta_n\right] d[\eta_n]$$

vectors of Grassmann variables (G.v.) η_n: $[\eta_{na}, \eta_{nb}]_+ = 0$

SECOND STEP: FOURIER–TRANSFORMATION

- interchange of Gaussian integrals with the average over the ordinary symmetric matrices
- performing the average

$\Rightarrow$ Fourier–transformation $\mathcal{F}$

$$P(H) \xrightarrow{\mathcal{F}} \mathcal{F}P(K)$$

- K contains z's and η's $\rightarrow$ see next transparency

CHARACTERISTIC FUNCTION

$$\mathcal{F}P(K) = \int P(H) \exp \imath \mathrm{tr}\, HK d[H]$$

inherits the rotation invariance from P

WISHART–LIKE MATRICES

- $N \times N$–dyadic ordinary matrix:

$$K = \sum_{m=1}^{p} z_m z_m^\dagger - \sum_{n=1}^{q} \eta_n \eta_n^\dagger = V^\dagger V$$

- $(p+q) \times (p+q)$–supermatrix:

$$B = VV^\dagger$$

- $(p+q) \times N$ rectangular supermatrix V
- for $\beta \in \{1,4\}$ 2's in the matrix dimensions of the quaternionic parts

Third Step: Duality relation

Crucial identity

$$\operatorname{tr} K^m = \operatorname{Str} B^m$$

for all $m \in \mathbb{N}$
$\rightarrow$ relation between **all** invariants in ordinary space and superspace

Duality relation

$$\mathcal{F}P(K) = \mathcal{F}P_0(\operatorname{tr} K, \operatorname{tr} K^2, \ldots) = \mathcal{F}P_0(\operatorname{Str} B, \operatorname{Str} B^2, \ldots) = \Phi_0(B)$$

Fourth Step: Dirac-distribution in superspace

Exchange of the integral over B by an integral over symmetric supermatrices

$$\Phi_0(B) \overset{?}{\longrightarrow} \Phi_0(\rho)$$

Heuristic approach

$$\Phi_0(B) = \int \Phi_0(\rho)\delta(\rho - B)d[\rho]$$

Two approaches for the step "**?**"

- Generalized Hubbard–Stratonovich transformation

$$\Phi_0(B) = \int \int \Phi_0(\rho) \exp[\imath \mathrm{Str}\, \sigma(\rho - B)]d[\rho]d[\sigma]$$

- Superbosonization formula

The generalized Hubbard–Stratonovich transformation

Integration over B, that is over the z's and η's

$$\int \Phi_0(\rho) \exp[\imath \mathrm{Str}\, \rho\sigma] \mathrm{Sdet}^{-N/\gamma}[\sigma - x^-] d[\sigma] d[\rho]$$

Supersymmetric Ingham–Siegel integral

$$I(\rho) = \int \exp[\imath \mathrm{Str}\, \rho\sigma] \mathrm{Sdet}^{-N/\gamma}[\sigma^+] d[\sigma]$$

- ordinary version for GUE: Fyodorov (2002)
- for $\beta = 2$: Guhr (2006)
- for $\beta \in \{1, 4\}$: Kieburg, Grönqvist, and Guhr (2008)

The generalized Hubbard–Stratonovich transformation

$$\int \Phi(B) d[z,\eta] \sim \int I(\rho)\Phi(\rho)\, d[\rho]$$

What is $I(\rho)$?

- heuristically: $I(\rho) \sim$ "$\mathrm{Sdet}^{\kappa}\rho$"
- more precisely: it consists of three essential parts
  1) determinant of the Boson–Boson block to the power κ
  2) Dirac–distributions at zero of the fermionic eigenvalues

exponent κ is function of N, p, q and β with $N \geq p$

Third part

Differential operator[1] D_{r_2} is an analog of the Sekiguchi–differential operator[2]

$$D_{r_2} = \frac{1}{\Delta(r_2)} \det \left[r_{n2}^{q-m} \left(\frac{\partial}{\partial r_{n2}} + (q-m)\frac{2}{\beta}\frac{1}{r_{n2}} \right) \right]$$

Remark

Fermion–Fermion block ρ_2 runs in a non-compact domain

[1] Kieburg, Grönqvist, Guhr[08]; [2] Okounkov, Olshanski[97]

The generalized Hubbard–Stratonovich transformation

$$\mathrm{Herm}\,(1,N) \rightarrow \left[\begin{array}{c|c} \mathrm{Herm}\,(1,p)>0 & pq\ G.v. \quad [pq\ G.v.]^* \\ \hline \begin{array}{c} -[pq\ G.v.]^\dagger \\ [pq\ G.v.]^T \end{array} & \mathrm{Herm}\,(4,q) \end{array}\right]$$

$$\mathrm{Herm}\,(2,N) \rightarrow \left[\begin{array}{c|c} \mathrm{Herm}\,(2,p)>0 & pq\ G.v. \\ \hline -[pq\ G.v.]^\dagger & \mathrm{Herm}\,(2,q) \end{array}\right]$$

$$\mathrm{Herm}\,(4,N) \rightarrow \left[\begin{array}{c|c} \mathrm{Herm}\,(4,p)>0 & \begin{array}{c} pq\ G.v. \\ [pq\ G.v.]^* \end{array} \\ \hline -[pq\ G.v.]^\dagger \quad [pq\ G.v.]^T & \mathrm{Herm}\,(1,q) \end{array}\right]$$

$\mathrm{Herm}\,(\beta,.)$: set of real symmetric ($\beta=1$), hermitian ($\beta=2$) and quaternionic selfadjoint ($\beta=4$) matrices
non-compact

Guhr[06]; Kieburg, Grönqvist, Guhr[08]; Kieburg, Sommers, Guhr[09]

$$\int \Phi(B)d[z,\eta] \sim \int \mathrm{Sdet}^{\kappa}\rho\Phi(\rho)\,d[\rho]$$

Remarks:

- same restriction $N \geq p$ and exponent κ as for the generalized Hubbard–Stratonovich transformation
- Fermion–Fermion block ρ_2 runs in a compact domain

$$\mathrm{Herm}\,(1,N) \quad\to\quad \left[\begin{array}{c|cc} \mathrm{Herm}\,(1,p)>0 & pq\ G.v. & [pq\ G.v.]^* \\ \hline -[pq\ G.v.]^\dagger & & \\ [pq\ G.v.]^T & \multicolumn{2}{c}{\mathrm{CU}\,(4,q)} \end{array}\right]$$

$$\mathrm{Herm}\,(2,N) \quad\to\quad \left[\begin{array}{c|c} \mathrm{Herm}\,(2,p)>0 & pq\ G.v. \\ \hline -[pq\ G.v.]^\dagger & \mathrm{CU}\,(2,q) \end{array}\right]$$

$$\mathrm{Herm}\,(4,N) \quad\to\quad \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{Herm}\,(4,p)>0} & \begin{array}{c} pq\ G.v. \\ [pq\ G.v.]^* \end{array} \\ \hline -[pq\ G.v.]^\dagger & [pq\ G.v.]^T & \mathrm{CU}\,(1,q) \end{array}\right]$$

$\mathrm{CU}\,(\beta,.)$: set of circular orthogonal (COE, $\beta=1$), unitary (CUE, $\beta=2$) and unitary-symplectic (CSE, $\beta=4$) ensembles
compact

Sommers[07]; Bunder et al.[07]; Littelmann, Sommers, Zirnbauer[08]; Kieburg, Sommers, Guhr[09]

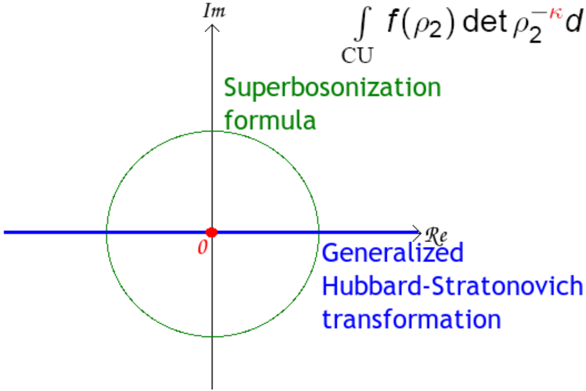
Generalized Hubbard–Stratonovich transformation

- $I(\rho) \sim$ "$\mathrm{Sdet}^{\kappa}\rho$" $\longrightarrow$ distribution in the Fermion–Fermion block
- non-compact fermionic integration

Superbozonization formula

- $I(\rho) \sim \mathrm{Sdet}^{\kappa}\rho \longrightarrow$ contour integral in the Fermion–Fermion block
- compact fermionic integration

$$\int_{\mathrm{CU}} f(\rho_2) \det \rho_2^{-\kappa} d[\rho_2] \sim D_{r_2}^{N-p} f(r_2)|_{r_2=0}$$



> Eigenvalue integrals in the Fermion–Fermion block: the superbosonization formula is the contour integral of the generalized Hubbard–Stratonovich transformation

Basile, Akemann[07]; Kieburg, Sommers, Guhr[09]

FOR FINITE N AND FOR ARBITRARY P

provided Φ_0 can be calculated

$\beta = 2$: all k–point correlations

$\beta \in \{1, 4\}$: level density (1–point correlation)

Summary

- extension of the generalized Hubbard–Stratonovich transformation for the orthogonal and unitary symplectic symmetry
- generalized Hubbard–Stratonovich transformation was made mathematically more rigorous
- comparison of the generalized Hubbard–Stratonovich transformation with the superbosonization formula

Thank you for your attention!

- M. Kieburg, H. Kohler and T. Guhr. “Integration of Grassmann variables over invariant functions on flat superspaces”, *J. Math. Phys.* **50**, 013528 (2008)
- M. Kieburg, J. Grönqvist and T. Guhr. “Arbitrary rotation invariant random matrix ensembles and supersymmetry: orthogonal and unitary–symplectic case” submitted to *J. Phys.* **A** (2008)
- M. Kieburg, H.-J. Sommers and T. Guhr. “Comparison of the superbosonization formula and the generalized Hubbard–Stratonovich transformation” submitted to *J. Phys.* **A** (2009)