Determinants and Pfaffians for random matrices and orthogonal polynomials

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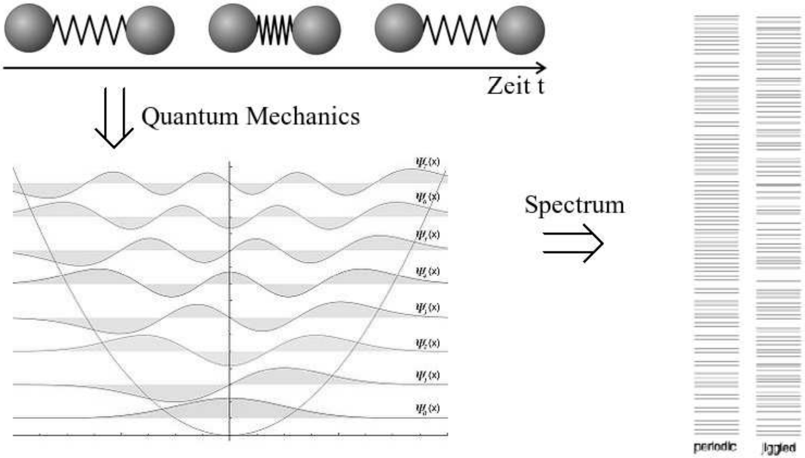
Outline

I. Introduction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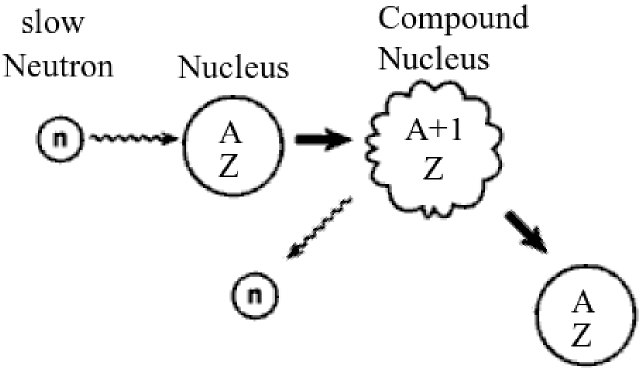
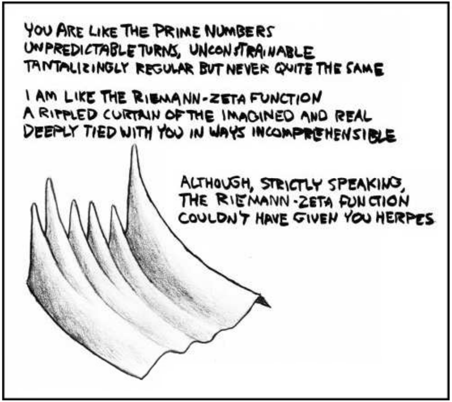
INTRODUCTION: RESONANCE SPECTRA



G. F. B. Riemann

$$\zeta(z) = \sum_{n=1}^{\infty} n^{z}$$



Introduction: Daily life

Questions

- What are the processes which create level repulsion or clustering in the spectra?
- What do they have in common?
- Are we able to compare all these spectra?

Answer to the third question

- unfolding procedure = scaling of all spectra on a normalized scale, for example on the scale of the local mean level spacing
- splitting into subspectra if good symmetries (mirror symmetries, spin, chirality) exist

INTRODUCTION: MAIN PROBLEMS

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Introduction: Main Problems

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Introduction: Objects of Interest

Eigenvalue correlations:

- What is the probability to find one level at x_1 and another at x_2? $\Rightarrow$ two–point correlation function

$$R_2(x_1,x_2)=\overline{\rho(x_1)\rho(x_2)}=\int P(H)\rho(x_1)\rho(x_2)d[H]$$

$$\text{with}\qquad \rho(x_i)=\sum_{n=1}^{N}\delta(x_i-E_n)$$

$$\Rightarrow R_2(x_1,x_2)\overset{|x_1-x_2|\to\infty}{\longrightarrow}\overline{\rho(x_1)}\;\overline{\rho(x_2)}$$

- What is the probability that the two neighboring levels have the distance s?
  $\Rightarrow$ level spacing distribution $p(s)$:
  no levels between x_1 and x_2

$$\Rightarrow p(s)\overset{s\to\infty}{\longrightarrow}0$$

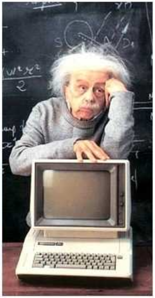
- etc.

II. Random Matrix Theory and Orthogonal Polynomials

real random variables only on the diagonal $\Longleftarrow$

$$H = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

eigenvalues are independently distributed



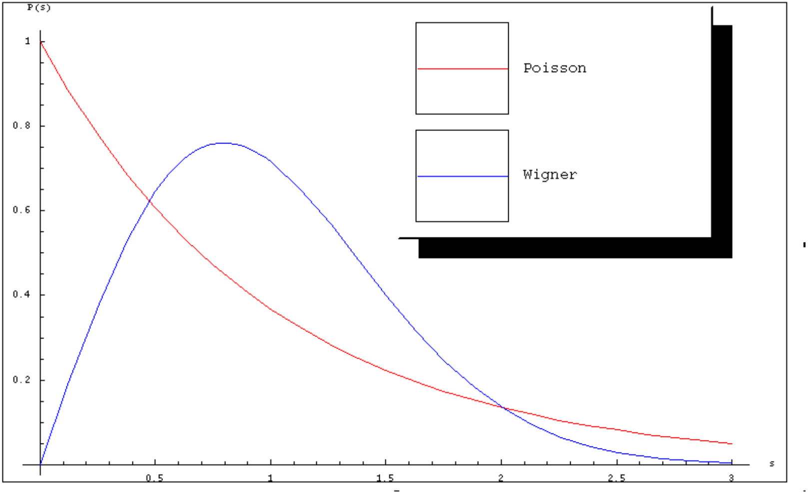
$$\exp\left[-\mathrm{tr}\, H^2\right]$$

$\Longrightarrow$ real random variables in each entry (symmetric)

$$H=H^T= \begin{bmatrix} * & & * \\ & \ddots & \\ * & & * \end{bmatrix}$$

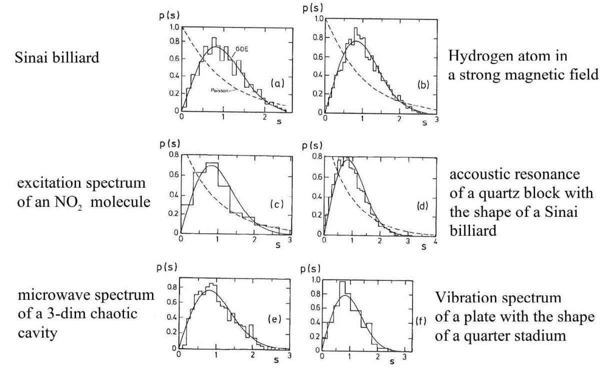
eigenvalues are correlated

Level Spacing Distribution for Random Matrices



Level Spacing Distribution for empirical data

Unfolded spectra



The mean value of the level spacing s is one for all spectra.
collected by Stöckmann (1999)

From an experimental pov: universal behavior of spectra
From a theoretical pov: universality of matrix ensembles

Starting point

$$Z_N(\kappa) = \int P(H) \prod_{i=1}^{k} \frac{\det(\kappa_{i2} + H)}{\det(\kappa_{i1} + H)} d[H]$$

- rotation invariant probability density function P
- $\kappa_{i1} = x_i + J_i$, $\kappa_{i2} = x_i - J_i$
- $J = 0 \quad \Rightarrow \quad Z_N(x) = 1$
- H is a $N \times N$ Hermitian or $2N \times 2N$ real symmetric or Hermitian selfdual matrix.

Relation to Green's and k-point correlation functions

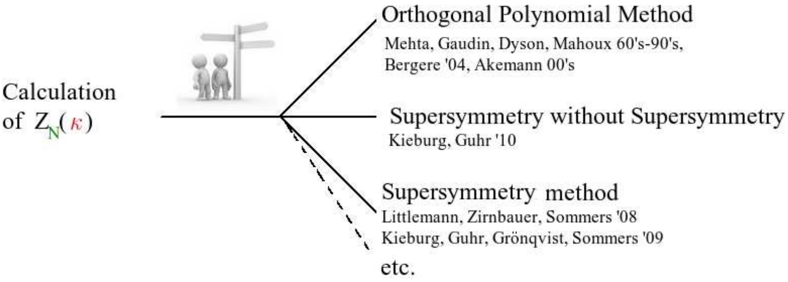
$$R_k(x) \sim \frac{\partial^k}{\partial J_1 \ldots \partial J_k} Z_N(\kappa)\Big|_{J=0}$$

Analytic Approach

Diagonalization

$$Z_N(\kappa) \sim \int P(E) \prod_{i=1}^{k} \frac{\det(\kappa_{i2} + E)}{\det(\kappa_{i1} + E)} |\Delta_N(E)|^{\beta} d[E]$$

- $H = UEU^\dagger$
- Dyson index $\beta \in \{1, 2, 4\}$
- Vandermonde determinant $\Delta_N(E) = \det[E_a^{b-1}]_{1 \leq a,b \leq N}$



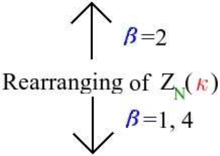
Calculation of $Z_N(\kappa)$
Orthogonal Polynomial Method
Mehta, Gaudin, Dyson, Mahoux 60's-90's,
Bergere '04, Akemann 00's
Supersymmetry without Supersymmetry
Kieburg, Guhr '10
Supersymmetry method
Littlemann, Zirnbauer, Sommers '08
Kieburg, Guhr, Grönqvist, Sommers '09
etc.

REQUIREMENT

Factorizing probability density function: $P(E) = \prod_{j=1}^{N} \widetilde{P}(E_j)$

TWO KINDS OF INTEGRALS

$$\int \prod_{j=1}^{N} \widetilde{P}(E_j) \prod_{i=1}^{k} \frac{\det(\kappa_{i2} + E)}{\det(\kappa_{i1} + E)} \Delta_N^2(E) d[E]$$



$$\int \prod_{j=1}^{N} g(E_{2j}, E_{2j+1}) \prod_{i=1}^{k} \frac{\det(\kappa_{i2} + E)}{\det(\kappa_{i1} + E)} \Delta_{2N}(E) d[E]$$

Kieburg, Guhr '10

Factorizing Ensembles: Results

Result for $\beta = 2$

$$Z_N(\kappa) \sim \frac{\det\left[\dfrac{Z_N(\kappa_{a1}, \kappa_{b2})}{\kappa_{a1} - \kappa_{b2}}\right]}{\det\left[\dfrac{1}{\kappa_{a1} - \kappa_{b2}}\right]}$$

Result for $\beta = 1,\ 4$

$$Z_N(\kappa) \sim \frac{\mathrm{Pf}\left[\begin{array}{c|c} (\kappa_{b2} - \kappa_{a2}) Z_{N-1}(\kappa_{a2}, \kappa_{b2}) & \dfrac{Z_N(\kappa_{b1}, \kappa_{a2})}{\kappa_{b1} - \kappa_{a2}} \\ \hline \dfrac{Z_N(\kappa_{a1}, \kappa_{b2})}{\kappa_{b2} - \kappa_{a1}} & (\kappa_{b1} - \kappa_{a1}) Z_{N+1}(\kappa_{a1}, \kappa_{b1}) \end{array}\right]}{\det\left[\dfrac{1}{\kappa_{a1} - \kappa_{b2}}\right]}$$

Orthogonal Polynomials ($\beta = 2$)

$$Z_N(\kappa_{a1}, \kappa_{b2}) \sim (\kappa_{a1} - \kappa_{b2}) \sum_{j=0}^{N-1} \frac{\widehat{p}_j(\kappa_{a1}) p_j(\kappa_{b2})}{h_j}$$

with p_n a polynomial of degree n and

$$\langle p_n | p_m \rangle_{\widetilde{P}} = \int p_n(x) p_m(x) \widetilde{P}(x) dx = h_n \delta_{nm}$$

$$\widehat{p}_j(y) = \left\langle \frac{1}{x-y} \middle| p_j \right\rangle_{\widetilde{P}} = \int \frac{p_j(x) \widetilde{P}(x)}{x-y} dx$$

$$Z_{N-1}(\kappa_{a2},\kappa_{b2}) \sim \frac{1}{\kappa_{a2}-\kappa_{b2}} \sum_{j=0}^{N-1} \frac{1}{h_j} \det \begin{bmatrix} q_{2j}(\kappa_{a2}) & q_{2j}(\kappa_{b2}) \\ q_{2j+1}(\kappa_{a2}) & q_{2j+1}(\kappa_{b2}) \end{bmatrix}$$

$$Z_{N}(\kappa_{a1},\kappa_{b2}) \sim (\kappa_{a1}-\kappa_{b2}) \sum_{j=0}^{N-1} \frac{1}{h_j} \det \begin{bmatrix} \widehat{q}_{2j}(\kappa_{a1}) & q_{2j}(\kappa_{b2}) \\ \widehat{q}_{2j+1}(\kappa_{a1}) & q_{2j+1}(\kappa_{b2}) \end{bmatrix}$$

$$Z_{N+1}(\kappa_{a1},\kappa_{b1}) \sim \frac{1}{\kappa_{a1}-\kappa_{b1}} \sum_{j=0}^{N-1} \frac{1}{h_j} \det \begin{bmatrix} \widehat{q}_{2j}(\kappa_{a1}) & \widehat{q}_{2j+1}(\kappa_{b1}) \\ \widehat{q}_{2j}(\kappa_{a1}) & \widehat{q}_{2j+1}(\kappa_{b1}) \end{bmatrix}$$

Skew-Orthogonal Polynomials ($\beta = 1,\ 4$)

$$\langle q_{2n} | q_{2m+1} \rangle_g = \int \det \left[\begin{array}{cc} q_{2n}(x_1) & q_{2n}(x_2) \\ q_{2m+1}(x_1) & q_{2m+1}(x_2) \end{array} \right] g(x_1, x_2) dx_1 dx_2 = h_n \delta_{nm}$$

$$\langle q_{2n} | q_{2m} \rangle_g = \langle q_{2n+1} | q_{2m+1} \rangle_g = 0$$

$$\widehat{q}_j(y) = \left\langle \frac{1}{x - y} \middle| q_j \right\rangle_g = \int \det \left[\begin{array}{cc} \dfrac{1}{x_1 - y} & \dfrac{1}{x_2 - y} \\ q_j(x_1) & q_j(x_2) \end{array} \right] g(x_1, x_2) dx_1 dx_2$$

III. Random Matrix Formulas for Orthogonal Polynomials

A BRIEF SURVEY

We know R_k when we calculate $Z_N(\kappa)$.

We know $Z_N(\kappa)$ when we calculate $p_j(x)$, $\widehat{p}_j(x)$, $q_j(x)$ and $\widehat{q}_j(x)$.

How can we caluculate the orthogonal and skew-orthogonal polynomials in an efficient way?

J. Gram E. Schmidt

- It is a recursive procedure.
- For large N it becomes cumbersome also with a computer.
- An explicit formula is more suitable.

- choosing an arbitrary basis $\{e_1, \ldots, e_N\}$
- normalize $e_1 \Rightarrow f_1$
- orthonormalize e_2 with respect to $f_1 \Rightarrow f_2$
- orthonormalize e_3 with respect to f_1 and $f_2 \Rightarrow f_3$
- $\vdots$

The Question

What is the basis in the dual space V' orthogonal to an arbitrary basis in V?

The Answer

Let $\{|e_1\rangle,\ldots,|e_N\rangle\}$ be an arbitrary basis in V and $\{\langle f_1|,\ldots,\langle f_N|\}$ be an arbitrary basis in V'. Then the orthonormalized dual basis to $\{|e_1\rangle,\ldots,|e_N\rangle\}$ is

$$\langle e_j'| = \frac{\det[\langle f_a|e_1\rangle \ldots \overbrace{\langle f_a|}^{j-\text{th coloumn}} \ldots \langle f_a|e_N\rangle]}{\det[\langle f_a|e_b\rangle]}\,.$$

The case $\beta = 2$

Identification

- V is the space of polynomials up to degree $N-1 \Rightarrow$ N-dimensional real space
- The monomials $m_j(x) = x^j$ build a basis in V.
- The scalar product identifies V with its dual space V' $\Rightarrow \langle F| : f(x) \mapsto \langle F|f\rangle_{\widetilde{P}} = \int F(x)f(x)\widetilde{P}(x)dx$
- The monomials $\langle m_j|$ build also a basis in V'.

Orthogonal Polynomials

p_j is orthogonal to m_l for $j > l \Rightarrow \langle p_j|$ are an orthogonal dual basis to m_j.

$$p_j(x) \sim \det[\langle m_a|m_0\rangle_{\widetilde{P}} \ \ldots \ \langle m_a|m_j\rangle_{\widetilde{P}} \ m_a(x)]_{0\leq a\leq j}$$

After some Algebraical Manipulation

$$p_j(x) \sim \int P(E)\det(x+E)\Delta_j^2(E)d[E]$$
$$\sim \int P(H)\det(x+H)d[H]$$

Szegö '59

Similar Manipulations for the Cauchy-Transform

$$\widehat{p}_j(x) \sim \int \frac{P(E)}{\det(x+E)}\Delta_{j+1}^2(E)d[E]$$
$$\sim \int \frac{P(H)}{\det(x+H)}d[H]$$

G. Akemann, A. Pottier '04

IDENTIFICATION

- V is the space of polynomials up to degree $2N-1 \Rightarrow$ $2N$-dimensional real space
- The monomials $m_j(x) = x^j$ build a basis in V.
- The symplectic form identifies V with its dual space $\Rightarrow \langle F|$:
  $f(x) \mapsto \langle F|f\rangle_g = \int \det \begin{bmatrix} F(x_1) & F(x_2) \\ f(x_1) & f(x_2) \end{bmatrix} g(x_1, x_2) dx_1 dx_2$
- The monomials $\langle m_j|$ build also a basis in V'.

SKEW-ORTHOGONAL POLYNOMIALS

q_{2j} and q_{2j+1} are orthogonal to m_l for $2j \geq l \Rightarrow \langle q_j|$ are an orthogonal dual basis to m_j.

$$q_{2j}(x) \sim \mathrm{Pf}\left[\begin{array}{c|c} 0 & m_b(x) \\ \hline -m_a(x) & \langle m_a|m_b\rangle_g \end{array}\right]_{0\leq a,b\leq 2j}$$

$$q_{2j+1}(x) \sim \mathrm{Pf}\left[\begin{array}{c|c|c} 0 & m_b(x) & m_{2j+1}(x) \\ \hline -m_a(x) & \langle m_a|m_b\rangle_g & \langle m_a|m_{2j+1}\rangle_g \\ \hline -m_{2j+1}(x) & \langle m_{2j+1}|m_a\rangle_g & 0 \end{array}\right]_{0\leq a,b\leq 2j-1} + c_j q_{2j}(x)$$

where c_j is an arbitrary constant.

AFTER SOME ALGEBRAICAL MANIPULATION

$$
\begin{aligned}
q_{2j}(x) &\sim \int \prod_{a=1}^{j} g(E_{2a}, E_{2a+1}) \det(x+E) \Delta_{2j}(E) d[E] \\
&\sim \int P(H) \det(x+H) d[H] \\
q_{2j+1}(x) &\sim \int \prod_{a=1}^{j} g(E_{2a}, E_{2a+1}) \\
&\times \det(x+E)[\operatorname{tr} E + x + c] \Delta_{2j}(E) d[E] \\
&\sim \int P(H) \det(x+H)[\operatorname{tr} H + x + c] d[H]
\end{aligned}
$$

For particular choices of g this formula was shown by Eynard '01, Ghosh and Pandey '02.

SIMILAR MANIPULATIONS FOR CAUCHY-TRANSFORM

$$\begin{aligned}
\widehat{q}_{2j}(x) &\sim \int \prod_{a=1}^{j+1} g(E_{2a}, E_{2a+1}) \det{}^{-1}(x+E) \Delta_{2j+2}(E) d[E] \\
&\sim \int \frac{P(H)}{\det(x+H)} d[H] \\
\widehat{q}_{2j+1}(x) &\sim \int \prod_{a=1}^{j+1} g(E_{2a}, E_{2a+1}) \frac{\operatorname{tr} E - c}{\det(x+E)} \Delta_{2j+2}(E) d[E] \\
&\sim \int \frac{P(H)(\operatorname{tr} H - c)}{\det(x+H)} d[H]
\end{aligned}$$

Conclusions

- simple formulas for the orthogonal and skew-orthogonal polynomials and their Cauchy-transform
- complicated integrals $\Rightarrow$ simple integrals

$\Rightarrow$ Other methods (e.g. supersymmetry method) benefit from such simplifications.

- applicable to a large class of matrix ensembles
- Results are also applicable to problems which have originally nothing to do with RMT.
- new results for $\beta = 1,\ 4$
- The two symmetries do not structurally differ in our approach.

Thank you for your attention!

- M. Kieburg and T. Guhr. “Derivation of determinantal structures for random matrix ensembles in a new way” *J. Phys.* **A 43**: 075201 (2009)
preprint: arXiv:0912.0654
- M. Kieburg and T. Guhr. “A new approach to derive Pfaffian structures for random matrix ensembles” *J. Phys.* **A 43**: 135204 (2010)
preprint: arXiv:0912.0658
- G. Akemann, M. Kieburg, and M. Phillips. “Skew-orthogonal Laguerre polynomials for chiral real asymmetric random matrices ” submitted to *J. Phys.* **A** (2010)
preprint: arXiv:1005.2983