"SUPERSYMMETRY WITHOUT SUPERSYMMETRY": DETERMINANTS AND PFAFFIANS IN RMT

Mario Kieburg

Universität Duisburg-Essen

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The setting

THE SETTING

Characteristic polynomial of a $N \times N$ matrix H is

$$q(x, H) = \det(H - x) \quad , \quad x \in \mathbb{C}$$

PROPERTIES

- q is a polynomial of order N in x
- q is invariant under similarity transformations

$$H \rightarrow THT^{-1} \quad \text{with } T \text{ invertible}$$

- roots of q with respect to x are the algebraic eigenvalues $\{E_1, \ldots, E_N\}$ of H

$\Rightarrow$ q only depends on x and $\{E_1, \ldots, E_N\}$

Average over rotation invariant matrix ensembles are interesting for:

- disordered systems
- quantum chaos
- matrix models in high energy physics
- quantum chromodynamics
- quantum gravity
- econo physics
- number theory
- Weyl's character formula
- theory of orthogonal polynomials

The setting

Setting

$$Z(\kappa,\lambda) \propto \int \frac{\prod_{n=1}^{k_2}\det(H-\kappa_{n2})\prod_{m=1}^{l_2}\det(H^\dagger-\lambda_{m2})}{\prod_{n=1}^{k_1}\det(H-\kappa_{n1})\prod_{m=1}^{l_1}\det(H^\dagger-\lambda_{m1})}d\mu(H)$$

+ rotation invariant, factorizable probability measure $d\mu$ with finite moments
(**Will be explained on the next few transparencies!**)

In general:

***H* does not have to be symmetric!**

The setting

Starting point (Hermitian $N \times N$ matrices)

$$Z_N(\kappa) = \int\limits_{\text{Herm}\,(N)} \prod_{j=1}^{k} \frac{\det(H - \kappa_{j2})}{\det(H - \kappa_{j1})} d\mu(H)$$

It is well known that this generating function exhibits a determinantal structure!
Baik, Deift, Strahov (2003); Grönqvist, Guhr, Kohler (2004); Borodin, Strahov (2005); Guhr (2006)

Similar determinantal structures for the k-point correlation function.
Mehta,Gaudin (1960)

For $N \times N$ Hermitian matrices:

- diagonalization $H = H^\dagger = UEU^\dagger$ with $U \in \mathrm{U}(N)$
  $E = \mathrm{diag}(E_1, \ldots, E_N)$
- factorizable probability measure $d\mu$:

$$d\mu(E) = \Delta_N^2(E) \prod_{j=1}^{N} d\widetilde{\mu}(E_j) \quad , \text{ with } E = \mathrm{diag}(E_1, \ldots, E_N)$$

Vandermonde determinant

$$\Delta_N(E) = \prod_{1 \le a < b \le N} (E_a - E_b) = \pm \det \overbrace{\left[{E_a}^{b-1} \right]}^{N} \Big\} N ,$$

The setting

Result

$$Z_N(\kappa) \quad \propto \quad \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \det\left[\frac{Z_N(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right]$$

with the Cauchy determinant

$$\sqrt{\mathrm{Ber}_{k/k}(\kappa)} = \frac{\Delta_k(\kappa_1)\Delta_k(\kappa_2)}{\prod\limits_{a,b=1}^{k}(\kappa_{a1}-\kappa_{b2})} = \pm\det\overbrace{\left[\frac{1}{\kappa_{a1}-\kappa_{b2}}\right]}^{k}\Big\}k\,,$$

$$\kappa = \mathrm{diag}\,(\kappa_1,\kappa_2) = \mathrm{diag}\,(\kappa_{11},\ldots,\kappa_{k1},\kappa_{12},\ldots,\kappa_{k2})$$

Orthogonal Polynomial Method

STARTING POINT

$$Z_N(\kappa) \propto \int \Delta_N^2(E) \prod_{a=1}^{N} d\widetilde{\mu}(E_a) \prod_{j=1}^{k} \frac{E_a - \kappa_{j2}}{E_a - \kappa_{j1}} ,$$

Vandermonde determinant is skew symmetric

$$\Delta_N(E) = \pm \det \left[\begin{array}{cccc} 1 & E_a & \cdots & E_a{}^{N-1} \end{array} \right]$$

$\Rightarrow$ allowing the construction of any basis for the polynomials up to order $N-1$

Baik, Deift, Strahov (2003)

Orthogonal Polynomial Method

Orthogonal Polynomials

- for $d\tilde{\mu}(x)$:

$$\int p_n(x)p_m(x)d\tilde{\mu}(x) = h_n\delta_{nm}$$

- for $d\tilde{\mu}(x)\prod_{j=1}^{k}\frac{x-\kappa_{j2}}{x-\kappa_{j1}}$:

$$\int \widetilde{p}_n(x)\widetilde{p}_m(x)d\tilde{\mu}(x)\prod_{j=1}^{k}\frac{x-\kappa_{j2}}{x-\kappa_{j1}} = \widetilde{h}_n\delta_{nm}$$

Cauchy Transform with respect to $d\tilde{\mu}(x)$

$$p_n^{(\mathrm{C})}(t) = \int \frac{p_n(x)}{x-t}d\tilde{\mu}(x) \text{ with } t \neq \mathbb{R}$$

RELATION BETWEEN BOTH BASES

For $k \leq n$:

$$\widetilde{p}_n(x) \propto \prod_{a=1}^{k} \frac{1}{x-\kappa_{a2}} \frac{\det\left[p_{n-k+b}^{(\mathrm{C})}(\kappa_{a1}) \;\middle|\; p_{n-k+b}(\kappa_{a2}) \;\middle|\; p_{n-k+b}(x)\right]}{\det\left[\; p_{n-k+c}^{(\mathrm{C})}(\kappa_{a1}) \;\middle|\; p_{n-k+c}(\kappa_{a2}) \;\right]}$$

with $1 \leq a \leq k$, $0 \leq b \leq 2k$ and $0 \leq c \leq 2k-1$

Uvarov 80's

GENERATING FUNCTION

Let $k \leq N$:

$$Z_N(\kappa) \propto \frac{\det\left[\ p^{(\mathrm{C})}_{N-k+b}(\kappa_{a1})\ \middle|\ p_{N-k+b}(\kappa_{a2})\ \right]}{\Delta_k(\kappa_1)\Delta_k(\kappa_2)}$$

with $1 \leq a \leq k$ and $0 \leq b \leq 2k-1$

DISADVANTAGES

- We have an artificial restriction $k \leq N$.
- The determinant is larger than the one for the k-point correlation function.
- We have to construct the orthogonal polynomials for the measure $d\widetilde{\mu}$.

Baik, Deift, Strahov (2003)

Supersymmetry Method

Supersymmetry Method

Starting point

$$Z_N(\kappa) = \int\limits_{\mathrm{Herm}\,(N)} \prod_{j=1}^{k} \frac{\det(H - \kappa_{j2})}{\det(H - \kappa_{j1})} d\mu(H)$$

In general

The measure $d\mu$ does not have to factorize for this method. One only needs the existence of its characteristic function and the rotation invariance!

Guhr (2006); Sommers, Zirnbauer, Littelmann (2007/08), Kieburg, Grönqvist, Sommers, Guhr (2008/09)

Supersymmetry Method

Gaussian integrals for determinants in the denominator

$$\frac{1}{\det(H-\kappa_{j1})} \sim \int \exp\left[-\imath\frac{\mathrm{Im}\,\kappa_{j1}}{|\mathrm{Im}\,\kappa_{j1}|} z_j^\dagger (H-\kappa_{j1}) z_j\right] d[z_j]$$

vectors of ordinary variables z_j: $[z_{ja}, z_{jb}]_- = 0$

Gaussian integrals for determinants in the numerator

$$\det(H-\kappa_{j2}) \sim \int \exp\left[\imath\eta_j^\dagger (H-\kappa_{j2})\eta_j\right] d[\eta_j]$$

vectors of Grassmann variables (G.v.) η_j: $[\eta_{ja}, \eta_{jb}]_+ = 0$

Integration

$$\int d\eta_i = 0 \qquad , \qquad \int \eta_i d\eta_i = \frac{1}{\sqrt{2\pi}}$$

The differentials $d\eta_i$ are also anti-commuting!

Integrations over arbitrary functions are defined by the power series in the Grassmann variables.

Outcome of both approaches (heuristic) for $k \leq N$

Map from ordinary space to superspace such that

$$Z_N(\kappa) \quad \sim \quad \int \mathrm{Sdet}^{-N}(\sigma-\kappa)d\widehat{\mu}(\sigma)$$

or in Fourier space

$$Z_N(\kappa) \quad \sim \quad \int \mathrm{Sdet}^{N}\rho\,\exp(-\imath\mathrm{Str}\,\rho\kappa)d\mathcal{F}\widehat{\mu}(\rho)$$

- $d\widehat{\mu}$ rot. inv. measure in superspace
- $d\mathcal{F}\widehat{\mu}(\rho)$ is "characteristic function" of $d\mu$ **and** $d\widehat{\mu}$
- σ and ρ are $(k+k)\times(k+k)$ supermatrices

SUPERDETERMINANT AND SUPERTRACE

- $$\sigma = \begin{bmatrix} \sigma_1 & \sigma_\eta^\dagger \\ \sigma_\eta & \sigma_2 \end{bmatrix}$$
- $$\mathrm{Sdet}\,\sigma = \frac{\det \sigma_1 - \sigma_\eta^\dagger \sigma_2^{-1} \sigma_\eta}{\det \sigma_2}$$
- $$\mathrm{Str}\,\sigma = \mathrm{tr}\,\sigma_1 - \mathrm{tr}\,\sigma_2$$

Hermitian supermatrices

- symmetric supermatrices with respect to the supergroup $U(p/q)$

$$\sigma \in \left[\begin{array}{c|c} \mathrm{Herm}\,(p) & [pq\mathrm{G.v.}]^\dagger \\ \hline pq\mathrm{G.v.} & \mathrm{Herm}\,(q) \end{array}\right] \quad , \quad \sigma = \sigma^\dagger$$

- diagonalization:

$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_\eta^\dagger \\ \sigma_\eta & \sigma_2 \end{bmatrix} \longrightarrow s = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \quad , \quad \sigma = UsU^\dagger$$

$\Rightarrow$ measure for the eigenvalues:

$$d\widehat{\mu}(\sigma) \longrightarrow \mathrm{Ber}_{p/q}(s) d\widehat{\mu}(s) + \textbf{b.t.}$$

“b.t.” are boundary terms (Efetov-Wegner-terms)

Supersymmetry Method

Definition

$$\sqrt{\mathrm{Ber}_{p/q}(s)} = \pm\frac{\Delta_p(s_1)\Delta_q(s_2)}{\prod\limits_{a=1}^{p}\prod\limits_{b=1}^{q}(s_{a1}-s_{b2})}$$

with $s = \mathrm{diag}\,(s_1, s_2) = \mathrm{diag}\,(s_{11}, \ldots, s_{p1}, s_{12}, \ldots, s_{q2})$

Determinantal structures ($p \leq q$)

$$\sqrt{\mathrm{Ber}_{p/q}(s)} \sim \det \overbrace{\begin{bmatrix} \dfrac{1}{s_{a1}-s_{b2}} \\ s_{b2}^{a-1} \end{bmatrix}}^{q} \begin{matrix} \}\,p \\ \}\,q-p \end{matrix}$$

mixes “Cauchy–terms” with “Vandermonde–terms”

Basor, Forrester[94] without considering the connection to supersymmetry;

Kieburg, Guhr[09/10] exhibiting the intimate relation to supersymmetry

Generating function

Let $k \leq N$:

$$Z_N(\kappa) \propto \int \frac{\det\left[\frac{1}{s_{a1}-s_{b2}}\frac{(s_{b2}-\kappa_{b2})^{N-1}}{(s_{a1}-\kappa_{a1})^{N+1}}\right]}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} d\widehat{\mu}(s) + \textbf{b.t.}$$

with $1 \leq a, b \leq k$

Disadvantages

- artificial restriction $k \leq N$
- mapping $d\mu$ to $d\widehat{\mu}$ (**Non-trivial task!**)
  Recently: compact formula for Hermitian, real symmetric and quaternionic selfdual matrices
- Efetov-Wegner-terms “b.t.” have a complicate structure for large k.
  Recently: construction of all “b.t.” for Hermitian matrices

Guhr (1991,2006)

“Supersymmetry without Supersymmetry”

STARTING POINT

$$Z_N(\kappa) \quad \propto \quad \int \Delta_N^2(E) \prod_{a=1}^{N} d\widetilde{\mu}(E_a) \prod_{j=1}^{k} \frac{E_a - \kappa_{j2}}{E_a - \kappa_{j1}} \, ,$$

Requirement:
The measure $d\mu$ for the matrices has to factorize to $d\widetilde{\mu}$!
All moments of $d\widetilde{\mu}$ and their Cauchy transform exist!

Kieburg, Guhr (2009/10)

Algebraic rearrangement

Multiplying

$$(I) = \prod_{a=1}^{N} \prod_{j=1}^{k} \frac{E_a - \kappa_{j2}}{E_a - \kappa_{j1}} \Delta_N(E)$$

by

$$(II) \quad = \quad \sqrt{\mathrm{Ber}_{k/k}(\kappa)}$$

ALGEBRAIC REARRANGEMENT

Result

$$
\begin{aligned}
(I)(II) &= \frac{\Delta_k(\kappa_1)\Delta_k(\kappa_2)\prod_{a=1}^{k}\prod_{b=1}^{N}(\kappa_{a2}-E_b)\Delta_N(E)}{\prod_{1\leq a,b\leq k}(\kappa_{a1}-\kappa_{b2})\prod_{a=1}^{k}\prod_{b=1}^{N}(\kappa_{a1}-E_b)} \\
&= \pm\sqrt{\mathrm{Ber}_{k/k+N}(\kappa, E)} \\
&= \pm\det\left[\begin{array}{c|c} \overbrace{\dfrac{1}{\kappa_{a1}-\kappa_{b2}}}^{k} & \overbrace{\dfrac{1}{\kappa_{a1}-E_b}}^{N} \\ \hline {\kappa_{b2}}^{a-1} & {E_b}^{a-1} \end{array}\right]\begin{array}{l}\}k \\ \}N\end{array}
\end{aligned}
$$

The variables κ_{a2} and E_b are the new fermionic eigenvalues!

EXPANSION OF THE SECOND VANDERMONDE DETERMINANT

$$Z_N(\kappa) \propto \int \frac{\sqrt{\mathrm{Ber}_{k/k+N}(\kappa, E)}}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \Delta_N(E) \prod_{a=1}^{N} d\widetilde{\mu}(E_a)$$

$$\propto \int \frac{\sqrt{\mathrm{Ber}_{k/k+N}(\kappa, E)}}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \prod_{a=1}^{N} {E_a}^{a-1} d\widetilde{\mu}(E_a)$$

$$\propto \frac{1}{\sqrt{\mathrm{Ber}_{k/k}(\kappa)}} \det \overbrace{\left[\begin{array}{c|c} \dfrac{1}{\kappa_{a1} - \kappa_{b2}} & \displaystyle\int \frac{E^{b-1} d\widetilde{\mu}(E)}{\kappa_{a1} - E} \\ \hline {\kappa_{b2}}^{a-1} & \int E^{a+b-2} d\widetilde{\mu}(E) \end{array}\right]}^{k \qquad\qquad N} \begin{array}{l} \}k \\ \}N \end{array}$$

LEIBNIZ EXPANSION OF THE DETERMINANT

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \ \det[A - BD^{-1}C]$$

IN OUR CASE

$$\begin{aligned} A_{ab} &= \frac{1}{\kappa_{a1} - \kappa_{b2}} \\ B_{ab} &= \int \frac{E^{b-1} d\widetilde{\mu}(E)}{\kappa_{a1} - E} \\ C_{ab} &= \kappa_{b2}{}^{a-1} \\ D_{ab} &= \int E^{a+b-2} d\widetilde{\mu}(E) \end{aligned}$$

yields a $k \times k$ determinant

Comparing the determinant entry for $k = 1$

$$\frac{Z_N(\kappa_{a1}, \kappa_{b2})}{\kappa_{a1} - \kappa_{b2}} \sim \frac{1}{\kappa_{a1} - \kappa_{b2}} - \sum_{1 \leq m,n \leq N} \int \frac{E^{m-1} d\widetilde{\mu}(E)}{\kappa_{a1} - E} D^{-1}_{mn} \kappa_{b2}{}^{n-1}$$

RESULT

$$Z_N(\kappa) \sim \frac{1}{\sqrt{\mathrm{Ber}\,_{k/k}(\kappa)}} \det\left[\frac{Z_N(\kappa_{a1},\kappa_{b2})}{\kappa_{a1}-\kappa_{b2}}\right]$$

There is no restriction $k \leq N$!

We have not used the explicit form of $d\widetilde{\mu}(E)$!
$\rightarrow$ These structures have to be true for other matrix ensembles!

General remarks and conclusions

TWO IMPORTANT QUESTIONS

Do all determinants and Pfaffians have the same origin?

If yes:

What are the conditions to find such structures?

For Pfaffians and determinants

- structures result from an algebraic construction
- the joint probability density has only to factorize and the integrals have to be finite, **no other requirements**

⇒ applicable to a broad class of ensembles

Some matrix ensembles yielding determinants

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability density $g(z)$
Hermitian ensemble [57, 31, 62, 32, 34, 35]	$\widetilde{P}\left(\mathrm{tr}\, H^m, m \in \mathbb{N}\right)$ $H = H^\dagger$	H	$P(x)\delta(y)$
circular unitary ensemble (unitary group) [37, 63, 14, 13, 38, 20, 64, 21]	$\widetilde{P}\left(\mathrm{tr}\, U^m, m \in \mathbb{N}\right)$ $U^\dagger U = 1\!\!1_N$	U and $U^\dagger$	$P\left(e^{\imath\varphi}\right)\delta(r-1)$
Hermitian chiral (complex Laguerre) ensemble [65, 66, 67, 7]	$\widetilde{P}\left(\mathrm{tr}(AA^\dagger)^m, m \in \mathbb{N}\right)$ A is a complex $N \times M$ matrix with $N \leq M$	$AA^\dagger$	$P(x)x^{M-N}\Theta(x)\delta(y)$
Gaussian elliptical ensemble [9, 10, 11, 36]; for $\tau = 1$ complex Ginibre ensemble	$\exp\left[-\frac{(\tau+1)}{2}\mathrm{tr}\, H^\dagger H\right] \times$ $\times \exp\left[-\frac{(\tau-1)}{2}\mathrm{Re}\;\mathrm{tr}\, H^2\right]$ H is a complex matrix; $\tau > 0$	H and $H^\dagger$	$\exp\left[-r^2\left(\sin^2\varphi + \tau\cos^2\varphi\right)\right]$
Gaussian complex chiral ensemble [12]	$\exp\left[-\mathrm{tr}\, A^\dagger A - \mathrm{tr}\, B^\dagger B\right]$ $C = \imath A + \mu B$ $D = \imath A^\dagger + \mu B^\dagger$ A and B are complex $N \times M$ matrices with $N \leq M$	CD and $D^\dagger C^\dagger$	$K_{M-N}\left(\frac{1+\mu^2}{2\mu^2}r\right) r^{M-N} \times$ $\times \exp\left(\frac{1-\mu^2}{2\mu^2} r\cos\varphi\right)$

General remarks and conclusions

Some matrix ensembles yielding Pfaffians

matrix ensemble	probability density P for the matrices	matrices in the characteristic polynomials	probability densities $g(z_1,z_2)$ and $\tilde{g}(z_1,z_2)$	probability density $h(z)$
real symmetric matrices [31, 24, 18]	$\widetilde{P}\,(\mathrm{tr}\,H^m, m\in\mathbb{N})$ $H=H^T=H^*$	H	$P(x_1)P(x_2)\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2-x_1)$	$P(x)\delta(y)$
circular orthogonal ensemble [4]	$\widetilde{P}\,(\mathrm{tr}\,U^m, m\in\mathbb{N})$ $U^\dagger U=\mathbb{1}_N$ and $U^T=U$	U and $U^\dagger$	$P(e^{\imath\varphi_1})P(e^{\imath\varphi_2})\times$ $\times\delta(r_1-1)\delta(r_2-1)\times$ $\times\Theta(\varphi_2-\varphi_1)$	$P(e^{\imath\varphi})\delta(r-1)$
real symmetric chiral (real Laguerre) ensemble [21, 32, 33, 34]	$\widetilde{P}\,(\mathrm{tr}(AA^T)^m, m\in\mathbb{N})$ A is a real $N\times M$ matrix with $\nu=M-N\geq 0$	AA^T	$P(x_1)P(x_2)\times$ $\times(x_1x_2)^{(\nu-1)/2}\times$ $\times\delta(y_1)\delta(y_2)\Theta(x_2-x_1)$	$P(x)\delta(y)x^{(\nu-1)/2}$
Gaussian real elliptical ensemble; for $\tau=1$ real Ginibre ensemble [10, 15, 35, 16, 36, 25] [37, 38, 23]	$\exp\left[-\frac{(\tau+1)}{2}\,\mathrm{tr}\,H^TH\right]\times$ $\times\exp\left[-\frac{(\tau-1)}{2}\,\mathrm{tr}\,H^2\right]$ $H=H^*$; $\tau>0$	H	$\prod\limits_{j\in\{1,2\}}\exp\left[-\tau x_j^2\right]\times$ $\times\sqrt{\mathrm{erfc}(\sqrt{2(1+\tau)}y_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2-x_1)+$ $+2\imath\delta^2(z_1-z_2^*)\Theta(y_1)]$	$\exp(-\tau x^2)\delta(y)$
Gaussian real chiral ensemble [9, 39]	$\exp\left[-\mathrm{tr}\,A^TA-\mathrm{tr}\,B^TB\right]$ $C=A+\mu B$ $D=-A^T+\mu B^T$ A and B are real $N\times M$ matrices with $\nu=M-N\geq 0$	CD	$\prod\limits_{j\in\{1,2\}}\exp\left[-2\eta_- z_j\right]\times$ $\times\lvert z_j\rvert^\nu\sqrt{f(2\eta_+ z_j)}\times$ $\times[\delta(y_1)\delta(y_2)\Theta(x_2-x_1)+$ $+2\imath\delta^2(z_1-z_2^*)\Theta(y_1)]$	$x^{\nu/2}\exp\left[-2\eta_- x\right]\times$ $\times K_{\nu/2}(2\eta_+ x)\delta(y)$

FOR MANY CHARACTERISTIC FUNCTIONS IN THE DENOMINATOR

- for $k_1 - k_2 - N \geq 0$:
  entries are either algebraic factors ("κ_{a1}^{b}", "$1/(\kappa_{a1} - \kappa_{b2})$"), many zeros or generating functions of ensembles consisting of one or two dimensional matrices

$\Rightarrow$ two regimes depending on the number of characteristic polynomials

General remarks and conclusions

Relation to orthogonal polynomial method

- supersymmetric structures are the ultimate reason for the Dyson-Mehta-Mahoux integration theorem
- determinants and Pfaffians lead to orthogonality and skew-orthogonality relations

$\Rightarrow$ orthogonal and skew-orthogonal polynomials and their Cauchy-transforms have simple expressions as matrix averages

For Pfaffians

Real and quaternionic ensembles are not distinguishable because their Pfaffians have the same origin!

OTHER APPLICATIONS

We use structures of supersymmetry without ever mapping our integrals onto superspace!

⇒Supersymmetry without supersymmetry

Thank you for your attention!

- M. Kieburg and T. Guhr. “Derivation of determinantal structures for random matrix ensembles in a new way” *J. Phys.* **A 43** 075201 (2010) preprint: arXiv:0912.0654
- M. Kieburg and T. Guhr. “A new approach to derive Pfaffian structures for random matrix ensembles” *J. Phys.* **A 43** 135204 (2010) preprint: arXiv:0912.0658
- G. Akemann, M. Kieburg, and M.J. Phillips. “Skew-orthogonal Laguerre polynomials for chiral real asymmetric random matrices” *J. Phys.* **A 43** 375207 (2010) preprint: arXiv:1005.2983